MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 102 No. 4

MEETING OF 1942 JUNE 12

Professor S. Chapman, President, in the Chair

The election by the Council of the following Fellow was duly confirmed:—

Albert Coleman, Warrant Officer I, Army Educational Corps, Queen Victoria School, Dunblane, Perthshire.

The following candidate was proposed for election as a Fellow of the Society, the name of the proposer from personal knowledge being appended:—

Edwin George Essery, M.R.C.S., L.R.C.P., St. Audry's Hospital, Melton, near Ipswich, Suffolk (proposed by H. E. Kersey).

Twenty-four presents were announced as having been received since the last meeting, including—

Photo-lithograph of Halley's Tidal Chart of the English Channel, 1701 (presented by the Royal Geographical Society).

The President referred to the death of the Rev. T. E. R. Phillips, Vice-President and ex-President of the Society, and asked the meeting to associate itself with the message of condolence which the Council was sending to Mr. Phillips's widow.

The President announced that the Council had elected the following as Associates of the Society:—

Dr. Robert R. McMath, Director of the McMath-Hulbert Observatory, Pontiac, Michigan, U.S.A.; and

Dr. Otto Struve, Director of the Yerkes Observatory, Williams Bay, Wisconsin, U.S.A.

MEETING OF 1942 JULY 10

Professor S. Chapman, President, in the Chair

The election by the Council of the following Fellows was duly confirmed:—

Major Alfred Marshall Graeme-Thom, M.C., Springfield, Cox Green Lane, Maidenhead, Berks.

Frederick E. Lewis, B.A., B.E., Ph.D., Lecturer in Civil Engineering, University College, Dublin; and 41 St. Kevin's Park, Dublin, S. 2.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Captain William Muir Ferguson Macfarlane, Director of the Fleetwood Navigational School, Fleetwood, Lancs. (proposed by J. R. Brown); and

Walter C. Rand, 58 Woodhouse Road, North Finchley, London, N. 12 (proposed by M. Davidson).

Forty-four presents were announced as having been received since the last meeting, including:—

B. F. Bawtree, Sun Tables for 0 to 2000 A.D.

The President announced that the Assistant Secretary, Miss Williams, was retiring through ill health after 22 years in the service of the Society; he tendered her the thanks of the meeting for her services and the good wishes of the Society in her retirement. He further announced that the Council had appointed as her successor Miss Edna Wadsworth, who had been for many years clerk to the Society.

The President further announced that the Ordinary Meeting in October would be devoted to a celebration of the anniversaries of Galileo, Newton and Halley.

The President also announced that there would be no George Darwin Lecture in 1942, Dr. Theodore Dunham (the invited lecturer) being unable to come over to England this year.

The Meeting then proceeded to a Discussion on the Structure and Rotation of Galaxies. The speakers were Dr. A. Hunter (Interstellar Absorption and the Scale of the Galaxy), Dr. H. A. Brück (The Structure of the Galaxy, with special reference to the neighbourhood of the Sun), Sir Arthur Eddington (The Rotation of the Milky Way) and Dr. H. Spencer Jones (The Rotation of Extra-galactic Nebulæ and their Internal Distribution of Mass).

ON THE INTERNAL CONSTITUTION OF THE STARS

F. Hoyle and R. A. Lyttleton

(Received 1941 June 9)

1. *Introduction.*—The central problem of stellar structure concerns the calculation of the luminosity L and radius R when the mass M and composition are known. The first step towards the solution came with the realisation that the equation of state of a perfect gas must be closely satisfied by the majority of stars and on this basis it was possible to shew that if the energy generation fulfilled fairly broad requirements the observed values of L, M and R could be reconciled together. For this to be so however it was found that most stars must contain large percentages of hydrogen. At first considerable reluctance was felt in accepting this result and it was regarded as preferable to suppose that the theoretical opacity coefficient was in defect of the actual amount. However it was shewn by Strömgren * in an exhaustive inquiry that this view could not be properly maintained and that the explanation must be ascribed to the composition of the stars. Since this demonstration the large hydrogen contents have been generally accepted.

The lack of definite knowledge of the mode of generation of energy however made it necessary to abandon any attempt to determine the luminosity and radius directly. Instead the observed values of L and R were used in the calculations, thereby imposing a surface condition on the stars, and it was only by introducing such a condition that it became possible to effect progress. Researches on the problem have hitherto proceeded along such lines. Accordingly, now that reliable estimates for the rate of generation of energy have been made † it is of interest to consider whether the use of such a condition is any longer really essential. That is, equipped with the more complete physical information now available, is it possible to solve the direct problem of finding L and R from a knowledge of the mass and composition? This question forms the subject of the present paper.

2. *The General Relations for Luminosity and Radius.*—The usual simplification of spherical symmetry of the star is adopted and the problem treated as one of equilibrium. If T and ρ denote the temperature and density, the coefficient of opacity at central distance r may be written

$$\frac{\alpha}{\tau} \cdot \rho(r)\{T(r)\}^{-3\cdot5},$$

where $\alpha = 3.10^{25}$ in c.g.s., etc. units and τ is a parameter that depends on the composition and the so-called guillotine factor. The equation of radiative equilibrium is then

$$\tfrac{4}{3}aT^3\frac{dT}{dr} = -\frac{\alpha}{c\tau} \cdot \rho^2 T^{-3\cdot5} \cdot L(r)/4\pi r^2, \qquad (1)$$

where a is Stefan's constant and c the velocity of light. $L(r)$ is the total amount of energy generated per second within distance r of the centre and may therefore be written

$$L(r) = 4\pi\int_0^r E\rho^2 T^\eta r^2 dr, \qquad (2)$$

the expression $E\rho^2T^\eta$ being the formula for the generation of energy per second per unit volume. This form for the theoretical rate of energy production has been given by Bethe

* B. Strömgren, *Ergebn. d. Exakt. Naturwiss.*, **16**, 465, 1937.

† G. Gamow and E. Teller, *Phys. Rev.*, **53**, 608, 1938; H. A. Bethe, *Phys. Rev.*, **55**, 434, 1939.

who has shewn that it is closely equivalent to the Gamow-Teller formula. In it the factor E depends on the composition, while the index η varies slightly with temperature. If η is given a definite value, E can be settled by the requirement that the carbon-nitrogen cycle generates approximately 100 ergs per sec. per gram when $\rho = 80$ gm. cm.$^{-3}$ and $T = 2.10^7$ degrees. For $\eta = 17{\cdot}25$, which is a convenient value to take, this leads to $E = 1{\cdot}43 \times 10^{-126}$. These particular values refer to a composition with 10 per cent. nitrogen and 35 per cent. hydrogen and would therefore apply accurately only to such conditions. For other concentrations E will usually be different, but it may reasonably be supposed that it will have an uncertainty of at most a factor of $(100)^{\pm 1}$ due to this cause and a change of this order turns out to have only a negligible effect on the calculated luminosity and radius. The approximation will therefore be made of taking E as constant both throughout any one star and from star to star. The index η varies nearly as $T^{-1/3}$ and it has been shewn by Bethe that it changes from about 18 for stars of mass similar to that of the Sun to about 16 for stars of mass about $15\odot$. Since however it is found that both L and R are very insensitive to changes in η by amounts of the order of unity, a good approximation is obtained by taking η as constant in any one star and from star to star, though this latter part of the assumption is not essential to the analysis. The question of possible variations in τ is considered below.

If at present the effect of radiation pressure is neglected, the mechanical equilibrium of the star requires

$$\frac{\mathfrak{R}}{\mu}\frac{d}{dr}(\rho T) = -G\rho r^{-2}\int_0^r 4\pi r^2 \rho\, dr = -G\rho r^{-2}M(r), \tag{3}$$

where $\mathfrak{R}$ is the gas constant, μ the mean molecular weight of the material and G the gravitation constant. In the present paper μ will be regarded as constant throughout any particular star.

From consideration of the form of (1), (2) and (3) it is now proposed to define four parameters r_c, λ, x and m in the following way:—

$$r_c^2 = \frac{4ac\tau}{3aE}\cdot\rho_0^{-4}T_0^{7\cdot5-\eta} \tag{4}$$

$$r_c^2 = x^2\cdot\frac{\mathfrak{R}T_0}{2G\mu\rho_0} \tag{6}$$

$$L = \tfrac{4}{3}\pi\lambda E\rho_0^2 T_0^{\eta} r_c^3 \tag{5}$$

$$m = \tfrac{4}{3}\pi\rho_0 r_c^3 \tag{7}$$

where ρ_0 and T_0 are the central values of ρ and T. It will be seen that r_c has dimensions of length and m of mass, while λ and x are pure numbers.

By eliminating T_0 and ρ_0 from these four equations it is possible to express L and r_c in terms of the functional parameters λ, x and m whose values or forms in terms of M and μ are at present unknown. To effect this elimination we temporarily write

$$l = \frac{3m}{4\pi}, \qquad \xi = \frac{\mathfrak{R}x^2}{2G\mu} \qquad \text{and} \qquad f = \frac{4ac\tau}{3aE}. \tag{8}$$

then (4), (6) and (7) lead to

$$\rho_0 = l^{-2}\xi^3 T_0^3 \tag{9}$$

and

$$T_0^{\eta+2\cdot5} = l^6\xi^{-10}f. \tag{10}$$

Then from (5) using (7), (9) and (10) in turn is obtained

$$L = E\lambda m\xi^{-1/3}f^{1/3}(l\xi^{-5/3}f^{1/6})^{\frac{8\eta+26}{2\eta+5}}$$

and inserting the values of l, ξ, f the luminosity takes the form

$$L = \tfrac{4}{3}\pi\lambda E\left(\frac{3m}{4\pi}\right)^{\frac{10\eta+31}{2\eta+5}}\left(\frac{2G\mu}{\mathfrak{R}x^2}\right)^{\frac{14\eta+45}{2\eta+5}}\left(\frac{4ac\tau}{3aE}\right)^{\frac{2\eta+6}{2\eta+5}}. \tag{11}$$

Similarly

$$r_c = \left(\frac{3m}{4\pi}\right)^{\frac{2\eta-7}{2\eta+5}} \left(\frac{2G\mu}{\Re x^2}\right)^{\frac{2\eta-15}{2\eta+5}} \left(\frac{3aE}{4ac\tau}\right)^{\frac{2}{2\eta+5}}. \tag{12}$$

These formulæ show that

$$L = p\lambda E^{-\frac{1}{2\eta+5}} \tau^{\frac{2\eta+6}{2\eta+5}} (\mu/x^2)^{\frac{14\eta+45}{2\eta+5}} m^{\frac{10\eta+31}{2\eta+5}}, \tag{13}$$

$$r_c = qE^{\frac{2}{2\eta+5}} \tau^{-\frac{2}{2\eta+5}} (\mu/x^2)^{\frac{2\eta-15}{2\eta+5}} m^{\frac{2\eta-7}{2\eta+5}}, \tag{14}$$

where p and q depend on natural constants only. Since η has a value of about $17(\pm 1)$, the foregoing equations shew clearly the result referred to earlier that both L and r_c are very little affected by even large changes in E.

Before the dependence of the luminosity on M can be found the dependence of λ, x and m/M has to be ascertained. In the following sections the values of these parameters are investigated and it is shewn that they are confined to fairly narrow limits over the whole range of possible values of μ and M.

3. It has been shewn by Strömgren that for stars of mass less than $5\odot$, in addition to its dependence on the composition, the factor τ depends on both ρ and T and we now consider briefly the effect of this. The quantity τ may be written in the form

$$\tau^{-1} = \frac{1 - X^2}{g} \cdot \left(\frac{\overline{Z^2}}{A}\right),$$

where A is the atomic weight, Z the nuclear charge, X the mass of hydrogen per gram of material, $(\overline{Z^2/A})$ is the average value of Z^2/A for the heavier elements weighted according to their abundance and g is the guillotine factor. The coefficient of opacity is therefore

$$\frac{a(1 - X^2)}{g} \cdot \left(\frac{\overline{Z^2}}{A}\right) \cdot \rho T^{-3\cdot 5}.$$

The factor g however varies with ρ and T and it has been found by Bethe that in the ranges of $\rho = 5$ to 100 gm. cm.$^{-3}$ and $T = 10^7$ to 3.10^7 degrees, Strömgren's numerical results can be well represented by writing $g \propto \rho^{1/2} T^{-3/4}$. Since also $g = 30$ for $\rho = 80$, $T = 2.10^7$ we may write

$$g = 10^6 \cdot \rho^{\frac{1}{2}} T^{-\frac{3}{4}}$$

and the coefficient of opacity therefore takes the form

$$a'(1 - X^2) \cdot (\overline{Z^2/A}) \cdot \rho^{1/2} T^{-11/4} = \frac{a'}{\tau'} \cdot \rho^{1/2} T^{-11/4} \tag{15}$$

defining τ' in this way, and where now $a' = 3.10^{19}$, provided X is not too near unity.

The dependence on $(\overline{Z^2/A})$ may be expected to lead to some variation in τ' from one star to another. The parameter g however is not subject to much change for stars of mass greater than $5\odot$ unless the hydrogen content is so large that scattering due to hydrogen exceeds that due to the heavier elements. For this to occur would require $1 - X^2$ to be of the order of 10^{-2}, or $X \sim 0{\cdot}99$, which is so impossibly high a proportion of hydrogen that this source of variation of g may be disregarded. In this paper the calculations for stars of small mass with moderate hydrogen contents are made on the basis of $\tau' \sim 1$.

If now the form (15) is written for the opacity in the equation of radiative equilibrium the corresponding appropriate definition of r_c is

$$r_c^2 = \frac{4ac\tau'}{3Ea'} \cdot \rho_0^{-\frac{7}{2}} T_0^{\frac{27}{4}-\eta} \tag{4'}$$

while the other three definitions (5), (6) and (7) are unaltered. An elimination similar to that made above now leads to

$$\rho_0 = l^{-2}\xi^3 T_0^3 \tag{9'}$$

and

$$T_0^{\eta+3\cdot5} = l^5 \xi^{-8\cdot5} f'. \tag{10'}$$

Hence we have

$$L = \tfrac{4}{3}\pi\lambda E\left(\frac{3m}{4\pi}\right)^{\frac{16\eta+53}{4\eta+7}}\left(\frac{2G\mu}{\Re x^2}\right)^{\frac{22\eta+81}{4\eta+7}}\left(\frac{4ac\tau'}{3E\alpha'}\right)^{\frac{4\eta+12}{4\eta+7}} \tag{11'}$$

$$r_c = \left(\frac{3m}{4\pi}\right)^{\frac{4\eta-13}{4\eta+7}}\left(\frac{2G\mu}{\Re x^2}\right)^{\frac{4\eta-27}{4\eta+7}}\left(\frac{3E\alpha'}{4ac\tau'}\right)^{\frac{4}{4\eta+7}}. \tag{12'}$$

For the purpose of comparison the expressions for L in the two cases may be written

$$L = p\lambda\tau(\mu/x^2)^7 m^5\{E^{-1}\tau(\mu/x^2)^{10}m^6\}^{\frac{1}{2\eta+5}} \tag{13}$$

$$L = p'\lambda\tau'(\mu/x^2)^{5\cdot5} m^4\{E^{-1}\tau'(\mu/x^2)^{8\cdot5}m^5\}^{\frac{5}{4\eta+7}} \tag{13'}$$

and the expressions for r_c may be written

$$r_c = q(\mu/x^2)m\{E^{-1}\tau(\mu/x^2)^{10}m^6\}^{-\frac{2}{2\eta+5}}$$

$$r_c = q'(\mu/x^2)m\{E^{-1}\tau'(\mu/x^2)^{8\cdot5}m^5\}^{-\frac{2}{2\eta+3\cdot5}}.$$

From these it is seen that the guillotine factor introduces some modification in the relation for L, but that the relation for r_c is practically unaffected.

The relation (13′) is of the type given by Bethe who however used approximate methods in which x, λ and m/M were assumed to be independent of M. In the following sections the degree to which these assumptions are satisfied is examined.

4. *The Equilibrium of the Central Regions of a Star.*—In order to find the parameters x, λ and m/M it is necessary to determine the temperature and density distribution in the star and this involves finding the type of thermodynamical equilibrium that must prevail. It will first be shewn that the whole star cannot be in radiative equilibrium and that the central regions must be in convective equilibrium. To prove this we proceed by two stages: (i) It is shewn that if for radiative equilibrium the effective polytropic index defined by $n = d\log\rho/d\log T$ once exceeds the value 3·25 then it must continue to increase with r, and (ii) That the solutions of equations (1) and (3) for ρ and T are such that n exceeds 3·25 very close to the centre whatever value (necessarily $\geqslant 1\cdot5$) n may take at $r=0$.

If (i) and (ii) are established it would then follow that no stable configuration exists entirely under radiative equilibrium, for (1) and (3) imply that $\rho^2 T^{-6\cdot5}$ must tend to a constant value as the surface of the star is approached, so that n must tend to 3·25 from below. Thus it will suffice to shew that n somewhere exceeds 3·25, for this means that the surface condition that ρ and T tend to zero simultaneously will be violated.

(i) From equations (1) and (3) and the definition of n we have

$$n+1 = \frac{16\pi ac\tau\mu G}{3\alpha\Re}\cdot\frac{M(r)}{L(r)}\cdot\frac{T^{6\cdot5}}{\rho^2},$$

where $M(r)$ is the mass within distance r of the centre, so that

$$\frac{\delta n}{n+1} = \frac{\delta M}{M} - \frac{\delta L}{L} + (2n-6\cdot5)\cdot\frac{(-\delta T)}{T}.$$

Hence if n exceeds 3·25 the last term in this is positive for an outward step, since δT is negative, and as far as this term is concerned n therefore continues to increase. To shew that the first two terms also conduce to increase n we have that

$$r^2\rho_r\delta r \cdot \frac{L}{\delta L} = \int_0^r r^2\rho dr \cdot \frac{\rho}{\rho_r} \cdot \left(\frac{T}{T_r}\right)^\eta,$$

where the suffix r refers to values at the upper limit of the integral. Now at distances less than r we have from the equilibrium equations

$$\rho T > \rho_r T_r \quad \text{and} \quad T > T_r.$$

Hence

$$\rho T^\eta > \rho_r T_r^\eta \quad \text{for any} \quad \eta > 1,$$

so that

$$r^2\rho_r\delta r \cdot \frac{L}{\delta L} > \int_0^r r^2\rho dr,$$

that is

$$\frac{\delta M}{M} > \frac{\delta L}{L},$$

since δL and δM are positive for an outward step.

Thus if n once exceeds 3·25 it will proceed to increase and the surface condition will not be satisfied.

(ii) It is proposed to solve for ρ and T by means of series in powers of the distance from the centre. For this purpose we put

$$T = 1 - \theta^2 + a_2\theta^4 + a_3\theta^6 + \ldots$$
$$\rho = 1 - n_0\theta^2 + b_2\theta^4 + b_3\theta^6 + \ldots$$

where θ is proportional to r, and T and ρ are here measured as fractions of the central values. Evidently n_0 is the central value of n and must be not less than 1·5. The arithmetic is slightly facilitated if η is here taken as 17·5. With the above expressions for T and ρ, n can also be derived as a series in θ^2.

With θ as variable instead of r equations (1) and (3) become

$$\frac{d}{d\theta}(T^{7\cdot5}) = -45\rho^2\theta^{-2}\int_0^\theta \rho^2 T^\eta\theta^2 d\theta,$$

$$\frac{d}{d\theta}(\rho T) = -6(n_0+1)\rho\theta^{-2}\int_0^\theta \rho\theta^2 d\theta.$$

For $n_0 = 1{\cdot}5$ the solution of these is found to be

$$T = 1 - \theta^2 + \tfrac{22}{5}\theta^4 - \tfrac{3187}{210}\theta^6 + \ldots$$
$$\rho = 1 - \tfrac{3}{2}\theta^2 - \tfrac{29}{10}\theta^4 + \tfrac{5937}{210}\theta^6 + \ldots$$

and

$$n(\theta) = \tfrac{3}{2}(1 + \tfrac{79}{6}\theta^2 + \tfrac{1327}{55}\theta^4 + \ldots).$$

Hence in this case n exceeds 3·25 at $\theta^2 = {\cdot}073$ about. For increasing values of n_0 it is found that the value of θ^2 at which n reaches 3·25 decreases rapidly, so that $n_0 = 1{\cdot}5$ is the most favourable condition for radiative equilibrium.

Thus for $n_0 = 2$, the corresponding series are

$$T = 1 - \theta^2 + \tfrac{26}{5}\theta^4 - \tfrac{21752}{1005}\theta^6 + \ldots$$
$$\rho = 1 - 2\theta^2 - \tfrac{12}{5}\theta^4 + \tfrac{208}{7}\theta^6 + \ldots$$

and

$$n(\theta) = 2(1 + \tfrac{69}{5}\theta^2 + \tfrac{85566}{1755}\theta^4 + \ldots).$$

Here n exceeds 3·25 before $\theta^2 = {\cdot}04$. For $n_0 = 2{\cdot}5$ this value is reached before $\theta^2 = {\cdot}02$

It may easily be verified that where n reaches 3·25, T and ρ have not departed far from their central values. Thus for the $n_0 = 1{\cdot}5$ case, $T > {\cdot}93$ and $\rho > {\cdot}88$ when $n = 3{\cdot}25$, while for larger values of n_0 their values are much closer to unity. The general expressions for the series for T, ρ and $n(\theta)$ for any η and n_0 have been obtained by the writers and it has been established that for any η in the range found by Bethe for this quantity the present conclusions follow.

It will be noticed that the foregoing argument proves not only that the whole star cannot be in radiative equilibrium but that the central regions must be under convective equilibrium. For if the central part were in radiative equilibrium n would continually increase outward and this would not permit the occurrence of convection anywhere, a necessary condition for which is that the value of n must somewhere fall to below the central value. On the other hand, whether convection currents occur or not in the outer part of the star, the distribution there must satisfy the condition for radiative equilibrium; for near the surface the density falls to so low a value that there can be insufficient material to enable an energy flux comparable with the luminosity to be transferred by convection. Accordingly these two considerations shew that the star must take up a configuration consisting of a central core in convective equilibrium surrounded by a radiative exterior. (The possibility of a star comprised of more than two zones alternately under convective and radiative equilibrium can be considered by numerical integrations and it is found that if convection occurs apart from in the central core it must extend to the surface, so that such a model could not satisfy the requirement of radiative equilibrium in the exterior.)

5. *Conditions for Convective Equilibrium.*—It is a necessary condition for convective equilibrium to be permanent that

$$\left|\frac{dT}{dr}\right|_{\text{ad.}} \leqslant \left|\frac{dT}{dr}\right|_{\text{rad.}}$$

where $\left|\frac{dT}{dr}\right|_{\text{ad.}}$ is the actual temperature gradient and $\left|\frac{dT}{dr}\right|_{\text{rad.}}$ is a gradient determined by (1) when the convective values of T and ρ are inserted everywhere in this equation. Thus $\left|\frac{dT}{dr}\right|_{\text{rad.}}$ becomes equal to the actual gradient when the equilibrium changes to radiative.

If we write

$$\psi = r/r_c$$

the equation defining the radiative gradient becomes

$$\frac{d}{d\psi}(T/T_0) = -(\rho/\rho_0)^2(T/T_0)^{-6{\cdot}5}\psi^{-2}\int_0^{\psi}(\rho/\rho_0)^2(T/T_0)^n\psi^2 d\psi. \tag{16}$$

The mechanical equation, which together with the relation $\rho/\rho_0 = (T/T_0)^n$ determines the actual values of ρ/ρ_0 and T/T_0 in the convective region, can be put in the form

$$\frac{d^2}{dr^2}(T/T_0) + \frac{2}{r}\frac{d}{dr}(T/T_0) + \frac{4\pi G\mu\rho_0}{\Re(n+1)T_0}(T/T_0)^n = 0. \tag{17}$$

By introducing a variable z of the dimensions of a number defined by

$$r = \left\{\frac{\Re(n+1)T_0}{4\pi G\mu\rho_0}\right\}^{\frac{1}{2}} z \tag{18}$$

the value of T/T_0 is the Emden function of z of index n. Thus the distribution of T/T_0 as a function of z is determined by (17).

As far as the first order in z is concerned the adiabatic gradient given by (17) is

$$\frac{d}{dz}(T/T_0) = -\tfrac{1}{3}z,$$

while the radiative gradient given by (16) to the same order is

$$\frac{d}{d\psi}(T/T_0) = -\tfrac{1}{3}\psi.$$

Hence these two gradients will be equal at and near the centre provided z and ψ are equal—that is, if

$$z = r/r_c. \tag{19}$$

With this relation between z and ψ (it requires a certain modification arrived at below), the value of x may now be determined from (6) which gives

$$x^2 = \frac{n+1}{2\pi} = \bar{x}^2 \quad \text{say.} \tag{20}$$

In order to determine the gradient given by (16) it is necessary to evaluate the integral appearing on the right, which may be written

$$L(z) = \int_0^z z^2 (T/T_0)^{n+2n} dz.$$

For $n = 1{\cdot}5$, $\eta = 17{\cdot}25$ the calculated values of $L(z)$ and $\left|\frac{dT}{dz}\right|_{\text{rad.}}$ are given in the second and third columns of the following table. The fourth and fifth columns give the actual gradient and the ratio of the radiative gradient to the adiabatic gradient.

z	$L(z)$	$\left\|\frac{dT}{dz}\right\|_{\text{rad.}}$	$\left\|\frac{dT}{dz}\right\|_{\text{ad.}}$	Ratio
0·0	·0	·0	·0	1·0
·2	·0026	·0655	·0663	0·981
·4	·0156	·1071	·1302	·823
·6	·0365	·1253	·1895	·661
0·8	·0549	·1248	·2424	·515
1·0	·0653	·1175	·2873	·409
1·2	·0694	·1129	·3231	·349
1·4	·0705	·1154	·3494	·330
1·6	·0707	·1282	·3661	·350
1·8	·0707	·1554	·3737	0·416
Limit	·0707	∞	·2033	∞

It is seen from this table that for a considerable range of z the radiative gradient is less than the adiabatic gradient and accordingly it can be decided that with $\psi = z$ the condition for convective equilibrium will not be satisfied and therefore that $\psi = z$ cannot correspond to a stable equilibrium state for the star. If however it is supposed that

$$z = \delta \,.\, \psi = \delta \,.\, r/r_c \tag{19'}$$

where δ is a disposable parameter, it becomes possible to choose δ so that the condition for stable convective equilibrium is satisfied. The corresponding value of the parameter x will then be

$$x = \bar{x} \,.\, \delta = \sqrt{\frac{n+1}{2\pi}} \cdot \delta. \tag{20'}$$

With this modified ψ, the radiative gradient now becomes from (16)

$$\frac{d}{dz}(T/T_0) = -\delta^{-2} \cdot (\rho/\rho_0)^2(T/T_0)^{-6\cdot5}z^{-2}\int_0^z (\rho/\rho_0)^2(T/T_0)^{\eta}z^2dz. \tag{21}$$

Thus the former value of $\left|\frac{dT}{dz}\right|_{\text{rad.}}$ is simply multiplied by δ^{-2}. On the other hand the introduction of δ does not affect the value of $\left|\frac{dT}{dz}\right|_{\text{ad.}}$ and hence the ratio of the gradients is altered by the factor δ^{-2}. By choosing δ to be less than unity the radiative gradient can be increased and reference to the above table of gradients for $\psi = z$ shows that for the case $n = 1\cdot5$ their ratio is least at about $z = 1\cdot4$ where the value of the ratio is $0\cdot330$. Hence the radiative gradient will always be at least as great as the adiabatic gradient if we take $\delta^2 = 0\cdot33$ or $\delta = \cdot575$. It does not follow from this however that the convective zone must end at $z = 1\cdot4$ with $\delta = \cdot575$ making the two gradients equal there, but it does follow that the change cannot occur at a value of $z > 1\cdot4$, since this would mean that in the convective region the radiative gradient would be less than the adiabatic gradient for a certain range of z in the neighbourhood of $z = 1\cdot4$. Thus the present argument shews that a factor $\delta < 1$ must be introduced in (19) for the central region to be in stable convective equilibrium and that the radius of this region $z_0 \leqslant 1\cdot4$.

The parameter λ may conveniently be determined at this stage before proceeding to consider further the value of δ. By means of (2) and (5) we have at once

$$\lambda = 3\int_0^{\text{surface}} (r/r_c)^2(\rho/\rho_0)^2(T/T_0)^{\eta}d(r/r_c).$$

Since no important contribution to $L(z)$ is made outside the central convective region we may therefore write, using (19′),

$$\lambda = 3\delta^{-3}\int_0^{\text{surface}} z^2(T/T_0)^{\eta+2n}dz = \bar{\lambda} \cdot \delta^{-3} \quad \text{say.} \tag{22}$$

For $n = 1\cdot5$, $\eta = 17\cdot25$, the value of $\bar{\lambda}/3$ has been seen to be $\cdot0707$. The integral has also been evaluated for other pairs of values of n and η. Thus for $n = 3$ and $\eta = 17\cdot25$ we find $\bar{\lambda}/3 = \cdot0621$, while for $n = 3$, $\eta = 16$ the value is $\cdot0674$. The value of $\bar{\lambda}$ is however probably restricted to an even narrower range than these figures indicate, for when n is near $1\cdot5$, η will be about 18, while for larger n the value of η will decrease to about 16. Thus the index $\eta + 2n$ may be subject to slight increase along the range of n. On the other hand the temperature decline for increasing n occurs more slowly. Hence it may be regarded as permissible to take $\bar{\lambda}$ as constant. For convenience the value

$$\bar{\lambda} = 0\cdot2$$

will be adopted in all the numerical examples considered later; this corresponds to a limiting value of $\cdot0667$ for the integral $L(z)$.

6. Returning to the question of the value of δ, we have been led to consider the problem of a star with a convective core of radius z_0 in which the distribution is given by the appropriate polytrope, while the remainder is in radiative equilibrium. The equations satisfied by ρ and T in the radiative portion may be written

$$\left.\begin{aligned} \frac{d}{dz}(T/T_0) &= -B(1-X^2)z^{-2}(\rho/\rho_0)^2(T/T_0)^{-6\cdot5} \\ \frac{1}{T_0}\frac{d}{dz}(\rho T) &= -Cz^{-2}\rho\chi(z) \end{aligned}\right\} \tag{23}$$

where

$$\chi(z) = \rho_0^{-1}\int_0^z \rho z^2 dz.$$

The quantities B and C are determined by the requirement that T and ρ and their derivatives shall be continuous at z_0 (at present unknown except that it is $\leqslant 1{\cdot}4$), so that they may be calculated at once for any selected z_0 from the appropriate polytropic table. It appears that the group of equations (23) has to be treated by numerical integration and the problem presented by them is to find the starting-point z_0 such that the values of T and ρ fall to zero simultaneously.

When the required value of z_0 is found the equality of the radiative and convective gradients at z_0 immediately fixes δ. To determine m we have

$$\begin{aligned} M &= 4\pi \int_0^{\text{surface}} \rho r^2 dr \\ &= 4\pi \rho_0 r_c^3 \delta^{-3} \int_0^{\text{surface}} \rho/\rho_0 \,.\, z^2 dz \\ &= 3m\bar{\chi} \,.\, \delta^{-3} \end{aligned}$$

where $\bar{\chi}$ is the limiting value of $\chi(z)$. Thus we may write

$$m/M = (3\bar{\chi})^{-1} \,.\, \delta^3 = (\bar{m}/M) \,.\, \delta^3. \qquad (24)$$

Thus the required integrations of (23) will determine δ and $\bar{\chi}$ and hence the parameters x, λ and m/M.

The solution of (23) has been obtained by Cowling * for the case $n = 1{\cdot}5$ when all the energy is regarded as generated within the convective core. The value of z_0 found was just less than 1·2. Neglecting the slight increase of $L(z)$ beyond the convective portion, the necessary integrations have been repeated by the writers and the distribution of T, ρ and χ reached shows close agreement with the results given by Cowling. The calculations lead to:

$$\begin{aligned} z_0 &= 1{\cdot}1924 \quad \text{so that} \quad \delta = {\cdot}59, \\ \bar{\chi} &= 3{\cdot}126 \quad \text{so that} \quad \bar{m}/M = {\cdot}10663, \\ z_s &= \text{surface value of } z = 6{\cdot}967. \end{aligned}$$

In view of the nature of the various approximations necessarily involved in the initial assumptions it is clear that the above calculations determine the parameters δ, $\bar{\lambda}$, $\bar{x}$ and $\bar{m}/M$ with sufficient accuracy for numerical purposes in the case of the extreme value of n of 1·5 for the convective region. The values for $n = 3$ in the convective region are obtained in a subsequent section.

7. *Numerical Comparison of the Theory for* $n = 1{\cdot}5$.—On the basis that radiation pressure can be neglected, the preceding analysis enables the values of the luminosity to be calculated. Expressed in terms of $\bar{x}$, $\bar{\lambda}$, $\bar{m}$, which denote the values of these parameters corresponding to $\psi = z$ (19), the luminosity becomes

$$L = [\overline{138}{\cdot}865]\delta^{-\frac{4\eta+12}{2\eta+5}} (\bar{x}^2)^{-\frac{14\eta+45}{2\eta+5}} \left(\frac{\bar{m}}{M}\right)^{\frac{10\eta+31}{2\eta+5}} \mu^{\frac{14\eta+45}{2\eta+5}} M^{\frac{10\eta+31}{2\eta+5}} \tau^{\frac{2\eta+6}{2\eta+5}}. \qquad (25)$$

As a definite example we shall consider $M = 5\odot$, $\mu = 1$, corresponding to $X = 1/3$. Also taking $\overline{Z^2/A} = 6{\cdot}0$, it has been calculated by Strömgren that the value of g is about 6, so that $\tau \sim 1$. Since τ appears raised to the power $\dfrac{2\eta+6}{2\eta+5}$ only, it need not be known with great accuracy. The values used in the numerical factor on the right of (25) are:

$$\begin{aligned} \bar{\lambda} &= 0{\cdot}2, & \eta &= 17{\cdot}25, \\ E &= 1{\cdot}43 \times 10^{-126}, & \alpha &= 3 \times 10^{25}, \end{aligned}$$

together with the standard values of G, $\mathfrak{R}$, a and c.

* *M.N.*, **96**, 42, 1935.

For $M=5\odot$, $\mu=1$ and the values of the parameters corresponding to $n=1\cdot5$ it is readily found that

$$L(5\odot,\ \mu=1)=2\cdot33\times10^{36}\text{ ergs per second.}$$

As already seen for values of M less than $5\odot$ the form (11′) must be used for L. From this an homology relation may be derived

$$L\propto M^{\frac{16\eta+53}{4\eta+7}}\mu^{\frac{22\eta+81}{4\eta+7}} \tag{26}$$

By means of this relation and with the above value for $M=5\odot$, $\mu=1$, it is found that

$$L(\odot,\ \mu=1)=2\cdot2\times10^{33}\text{ ergs per second.}$$

This result is evidently in satisfactory agreement with the value of the solar luminosity obtained from observation.

Returning to the calculations for $M=5\odot$, $\mu=1$, equation (10) gives the central temperature and (9) then gives the central density. It is found that

$$T_0=3\cdot04\times10^7\text{ degrees,}$$
$$\rho_0=6\cdot67\text{ gm. cm.}^{-3}.$$

The radius may be found either directly from (12) or now more simply by calculating the unit of length associated with z. This unit is given by (18) with n put equal to 1·5. With the present values of T_0 and ρ_0 this gives

$$\text{unit of } z=\left\{\frac{5\mathfrak{R}T_0}{8\pi\mu G\rho_0}\right\}^{1/2}=3\cdot37\times10^{10}\text{ cm.}$$

Since the star extends to $z_s=6\cdot967$ the value of the radius is $2\cdot35\times10^{11}$ cm., which is about 3·4 times the solar radius.

For stars of mass less than $5\odot$ an homology for the radius is obtained from (12′) and has the form

$$R\propto M^{\frac{4\eta-13}{4\eta+7}}\mu^{\frac{4\eta-27}{4\eta+7}}.$$

The zero point of this relation is $R=3\cdot4R_\odot$ for $M=5\odot$, $\mu=1$.

8. *Effect of Radiation Pressure.*—At this stage we consider what changes are likely to be produced in the foregoing results when β the ratio of the gas pressure to the total pressure is appreciably less than unity.

The earlier formulæ can be modified very simply to allow for a great deal of the effect of radiation pressure if the assumption of β constant throughout the star is made. (This will be termed assumption I.) For then the equation of state takes the form

$$\text{Total pressure}=\frac{\mathfrak{R}}{\beta\mu}\cdot\rho T,$$

so that μ has everywhere to be replaced by $\beta\mu$. When the value to be assigned to β is known the change in luminosity due to this part of the necessary modification is therefore readily made. Before discussing the determination of β we may consider briefly how far assumption I is likely to be fulfilled in the extreme cases of relatively large radiation pressure, β near zero, and small radiation pressure, β near unity.

(i) $\beta\sim0$. When radiation pressure is taken into account the material may be regarded as behaving with an effective ratio of specific heats γ given by

$$\gamma=(-\beta^2-8\beta+\tfrac{32}{3})/(8-7\beta) \tag{27}$$

when the ordinary ratio of specific heats $\Gamma=5/3$. Accordingly for constant β the value

of the corresponding index $n=(\gamma-1)^{-1}$ changes from 1·5 to 3 as β goes from 1 to 0. Thus when β is small the index n must be near 3. But when n lies near 3 the whole star will be distributed in close accordance with the polytrope of index 3 and then β is automatically constant throughout the star. Thus for large radiation pressure compared with gas pressure it appears that the approximation of regarding β as constant is then valid. Since the proportion of radiation pressure is found to increase with the mass the present assumption becomes an increasingly good approximation and therefore the derivation of the luminosities of stars of very large mass on the basis of β constant will not be expected to lead to any important source of error.

(ii) $\beta \sim 1$. In this case I is not so acceptable an assumption, but nevertheless it may probably be used as a first approximation without introducing serious error. For in the configuration with $n=1{\cdot}5$ in the convective core, the radiative exterior is such that the effective index differs only slightly from 3 for values of $z > 2{\cdot}5$ and therefore this part of the star could be closely represented by taking $n=3$, which would imply constant β in this portion. On the other hand in the interior the ratio of gas pressure to radiation pressure is given by

$$\frac{3\mathfrak{R}\rho}{\mu a T^3}=\frac{\beta}{1-\beta}=\frac{\beta_0}{1-\beta_0}\cdot\frac{\rho T_0^{\,3}}{\rho_0 T^3} \tag{28}$$

where β_0 is the central value of β. From this equation the way in which β/β_0 changes with z may be estimated by using the values of ρ and T given by the configuration of zero radiation pressure. Since in this case n takes its smallest value it seems likely that the procedure will indicate greater variation in β than occurs for cases when the index exceeds 1·5 in the convective core. The value of $\rho T_0^{\,3}/\rho_0 T^3$ is found to increase from 1 at the centre to about 2·1 at $z=2{\cdot}5$. For $\beta_0=0{\cdot}8$ it then follows that β/β_0 changes from 1 to 1·12 at most, while for higher values of β_0 the range of variation is less than this. Thus it would appear that even for stars of moderate mass the approximation of taking β constant throughout the whole star is a passably satisfactory working assumption short of making detailed integrations taking proper account of radiation pressure.

There is also to be considered however the modification of the temperature and density distribution in the star due to this change in effective index. On the assumption of constant β, this would lead to a convective region in which $n > 1{\cdot}5$ and this in turn would give different values for the various parameters involved in the formulæ for L and R. The values of these parameters for $n=3$ in the central region are obtained in the next section. For intermediate values of n the parameters δ, $\bar{x}^2$ and $\bar{m}/M$ will be obtained by linear interpolation (with n) between the end values. This will be termed assumption II.

The determination of β may be made at once from equation (9) with the use of (20′), (24) and (28). It is found that

$$\frac{\beta_0^{\,4}}{1-\beta_0}=1{\cdot}26(\odot/M)^2\mu^{-4}\bar{\chi}^2\left(\frac{\gamma}{\gamma-1}\right)^3, \tag{29}$$

Equation (29) agrees closely in form with the well-known equation for β, though the present result allows for the possible changes in γ and $\bar{\chi}$ which as has been seen themselves depend on β. The dependence of γ on β is given by (27), while that of $\bar{\chi}$ on γ and hence on β is given in the present discussion by assumption II, so that when β is given both γ and $\bar{\chi}$ are known. Thus (29) enables the value of M corresponding to any μ and β to be calculated.

An interesting result of the numerical values for $n=1{\cdot}5$ and $n=3$ (determined later) is that the factor $\bar{\chi}^2(\gamma/\gamma-1)^3$ occurring in (29) changes only by a comparatively small amount along the whole range of β. At $\beta=1$ its value is $1{\cdot}53\times10^2$, while at $\beta=0$ it is $2{\cdot}59\times10^2$.

In the following table the values of $M\mu^2/\odot$ determined by (29) are given for a few values of β:—

β	$M\mu^2/\odot$	β	$M\mu^2/\odot$
0·1	1684·4	0·6	28·22
·2	389·2	·7	17·55
·3	158·6	·8	10·68
·4	81·01	·9	5·756
0·5	46·40	0·95	3·562

9. *Opacity by Electron Scattering.*—In addition to the photoelectric effect there is a second source of opacity in the scattering of radiation by free electrons. This latter process if acting alone would produce an opacity coefficient $0{\cdot}2(1+X)$ and it is easily verified by means of the numerical results already available that for stars of small mass this contribution to the total opacity is negligible. On the other hand as the mass increases the relative importance of electron scattering increases until for stars of large mass it appears that by far the main source of opacity is from the latter effect.

For, in a similar way to that used in deriving (29), it is found with some slight approximation in the indices on the right-hand side that

$$\frac{a(1-X^2)\rho_0 T_0^{-3\cdot5}}{0{\cdot}2(1+X)} = 1{\cdot}0(\odot/M)^2\bar{\chi}^2(\mu\beta)^{-3}\left(\frac{\gamma}{\gamma-1}\right)^3(1-X). \tag{30}$$

The values of M for which the two sources are equal for three representative values of μ have been estimated from this formula, which is of course only an approximation since it does not allow for the opacity arising from electron scattering. The results are as follows:—

$\mu=1/1{\cdot}85,\quad X=0{\cdot}9$ $\quad M=11\odot$ $\qquad$ $\mu=1,\quad X=0{\cdot}33$ $\quad M=25\odot$ $\qquad$ $\mu=2,\quad X=0$ $\quad M=24\odot$

Thus the two sources become comparable long before the mass attains very large values. It is clear that the introduction of both effects into the radiative equation would render it particularly difficult to handle, and to obviate this the photoelectric law alone will be used when it gives the larger opacity coefficient and the electron scattering law alone when it gives the larger. This will be referred to as assumption III. When the two sources are nearly equal it may be estimated that the effect of this assumption will be to increase the calculated luminosity by a factor near 2 since L varies practically as α^{-1}. It is possible that as a first approximation the effect of the extra source of opacity could be allowed for by increasing α by a constant amount, just as radiation pressure may be dealt with by increasing $\mathfrak{R}$ by a constant factor. It may also be remarked that as with assumptions I and II, the present approximation enables both ends of the mass-luminosity curves to be fixed, corresponding to large and small masses, and no cumulative source of error is thereby introduced.

On this basis the equation of radiative equilibrium for large masses is now

$$\tfrac{4}{3}acT^3\frac{dT}{dr} = -0{\cdot}2(1+X)\rho r^{-2}\int_0^r E\rho^2 T^\eta r^2 dr, \tag{31}$$

while the mechanical equation is unaltered from (3) except for the introduction of β which will usually differ considerably from unity for stars of great mass. To find the corresponding relations for the luminosity and radius we now make the definitions

$$r_c^2 = \frac{20ac}{3E(1+X)}\rho_0^{-3}T_0^{4-\eta} = x^2\frac{\mathfrak{R}T_0}{2G\mu\beta\rho_0}$$

while (5) and (7) are unaltered so that the equation (29) for β is unchanged. Writing

$$l=\frac{3m}{4\pi}, \qquad \xi=\frac{\Re x^2}{2G\mu\beta} \qquad \text{and} \qquad f=\frac{20ac}{3E(1+X)},$$

the relation for the luminosity without approximation in the indices is found to be

$$L=\tfrac{4}{3}\pi\lambda f E l^3\xi^{-4}. \tag{32}$$

Since fE is independent of E we therefore arrive at the remarkable result that when the opacity is due wholly to electron scattering the luminosity is independent of E and η (except in so far as $\bar{\lambda}$ may be slightly dependent on η) and this means that L is practically independent of the process of energy generation for reactions that can be represented in the form $E\rho^2T^\eta$.

The relation for the radius in this case is

$$r_c=l^{\frac{\eta-1}{\eta+3}}\xi^{-\frac{\eta-4}{\eta+3}}f^{-\frac{1}{\eta+3}}. \tag{33}$$

In order to find the values of the parameters occurring in (32) and (33) a procedure analogous to that given previously for the case of photoelectric opacity must be carried out. It is readily shewn to begin with that the central regions must be in convective equilibrium. The outer part must again be in radiative equilibrium, but now the effective index rises to 3 at the surface. When β is small the index in the convective region will not be far from 3 and it is therefore clear that for stars of very large mass the distribution will approach closely to that of the polytrope $n=3$, though this limit can never actually be reached.

From (31) the radiative gradient is determined by

$$\frac{d}{dz}(T/T_0)=-\delta^{-2}(\rho/\rho_0)(T/T_0)^{-3}z^{-2}\int_0^z(\rho/\rho_0)^2(T/T_0)^\eta z^2dz. \tag{21'}$$

Here as before z is defined by (18) and δ by (19'), while x also is unchanged in form and is given by (20'). If in accordance with assumption I the constant value of the index in the convective region is denoted by n_0, the radiative gradient becomes

$$\frac{d}{dz}(T/T_0)=-\delta^{-2}\,.\,(T/T_0)^{n_0-3}z^{-2}L(z).$$

The term $L(z)$ reaches its limiting value $L(z_s)$ say, before the ratio of the radiative and adiabatic gradients becomes least for any value of n_0 from 1·5 to 3, so that since the adiabatic gradient is $-\chi(z)/z^2$, we have outside the energy-producing regions

$$\left|\frac{dT}{dz}\right|_{\text{rad.}}\div\left|\frac{dT}{dz}\right|_{\text{ad.}}=\delta^{-2}\,.\,(T/T_0)^{n_0-3}L(z_s)/\chi(z). \tag{34}$$

If n_0 is less than 3 this ratio increases very rapidly near the surface of the polytrope, so that its minimum value must occur at a value $z_0<z_s$ and accordingly the radiative region must begin at a value $z\leqslant z_0<z_s$. If however $n_0=3$, the ratio of the gradients depends only on $\chi(z)^{-1}$ and this takes its minimum at the surface. For $n_0=2{\cdot}5$ the curve of the ratio of the gradients is very flat in the neighbourhood of its minimum and this property must become more marked as n_0 tends to 3, so that no important change in δ will be caused through the change to radiative equilibrium occurring nearer the centre than the minimum. (In the $n=1{\cdot}5$ case for example the change arising in this way was from ·57 to ·59 only.)

Thus in the limiting case $n_0 = 3$, δ may be decided with sufficient accuracy by making the gradients equal at the surface; the values arrived at are then:

$$z_0 = z_s = 6{\cdot}897 \quad \text{so that} \quad \delta = {\cdot}18$$
$$\chi = 2{\cdot}018 \quad \text{so that} \quad \bar{m}/M = {\cdot}16516$$

The dependences of the parameters on δ given by (20′), (22) and (24) are unaltered.

10. In order to obtain the mass-luminosity curves the values of the parameters for values of n intermediate between 1·5 and 3 will be estimated by interpolating linearly with n for δ, $\bar{x}^2$ and $\bar{m}/M$. The value of $\bar{\chi}$ required in (29) is then given by (24). Obtained in this way the mass-luminosity curves will evidently be well determined for values of n near 1·5 and 3·0. The following table shows the values of the parameters derived by interpolation:—

β	n	$\bar{x}^2$	$\bar{m}/M$	$\bar{\chi}$	δ
0·0	3·0	·6366	·1652	2·018	·18
·1	2·855	·6136	·1595	2·090	·22
·2	2·720	·5920	·1542	2·161	·26
·3	2·592	·5716	·1492	2·234	·30
·4	2·468	·5520	·1444	2·308	·34
·5	2·348	·5328	·1397	2·386	·38
·6	2·227	·5135	·1350	2·469	·42
·7	2·099	·4933	·1300	2·564	·46
·8	1·957	·4706	·1245	2·679	·51
·9	1·777	·4420	·1174	2·838	·55
0·95	1·658	·4231	·1128	2·955	·57
1·0	1·5	·3979	·1066	3·126	·59

(i) *Electron Scattering Opacity.*—Inserting the numerical values of the various constants involved in (32), the luminosity when the opacity is due to electron scattering only becomes

$$L = [\overline{64}{\cdot}760]\delta^{-2}\bar{\lambda} \,.\, (\bar{m}/M)^3 \bar{x}^{-8} \beta^4 \mu^4 (1+X)^{-1} M^3. \qquad (35)$$

In equation (34), remembering that $L(z_s) = \bar{\lambda}/3$, it is seen that δ and $\bar{\lambda}$ appear in the same way as in the expression (35) for L, that is in the product $\delta^{-2}\bar{\lambda}$. Now as n_0 approaches 3 the term in T/T_0 in (34) is practically unity and hence the quantity $\delta^{-2}\bar{\lambda}$ must be nearly equal to $3\bar{\chi}$. Accordingly the luminosity is almost entirely independent of η. In finding δ in the foregoing table the value $\bar{\lambda} = 0{\cdot}2$ has been used in conformity with the discussion following equation (22). In the accompanying figure the upper portions of the *M-L* curves have been computed by means of formula (35).

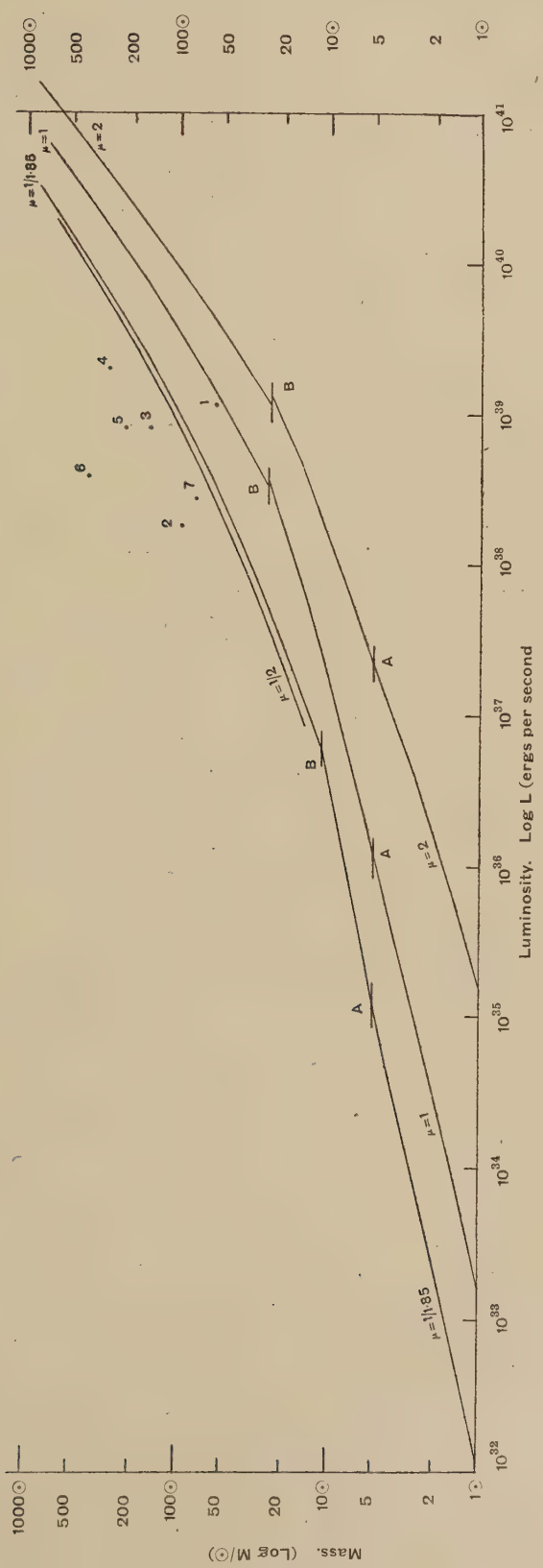
The suggestion that the well-known difficulty presented by the luminosities of the Trumpler stars could be overcome by attributing the opacity to electron scattering has been made independently by Tuominen.* The positions of these stars in the *M-L* figure are indicated by dots and it is seen that closest agreement is obtained if these bodies are regarded as constituted solely of hydrogen, that is $\mu = 0{\cdot}5$. The curve for $\mu = 0{\cdot}5$ shows that if the whole of the discrepancy is associated with the luminosity the average value of the quantity

$$\mathrm{m}_{\text{observed}} - \mathrm{m}_{\mu=0{\cdot}5}$$

is 1·1 magnitudes, whereas the investigation by Tuominen led to a value 1·8 magnitudes for this difference. On the other hand for Kuiper's estimated minimum values of the masses of these stars the average value of this difference is reduced to 0·6 magnitude.

(ii) *Photoelectric Opacity.*—In this case the luminosity for stars of mass not less than $5\odot$ is determined by equation (11) which with the numerical values of the constants inserted becomes

* *M.N.*, 100, 86, 1939.



Mass-Luminosity Curves.

$$L = [\overline{13}8{\cdot}865]\delta^{-2-\zeta}(\bar{m}/M)^{5+3\zeta}\bar{x}^{-14-10\zeta}(\beta\mu)^{7+5\zeta}M^{5+3\zeta} \tag{36}$$

where $\zeta = 2/(2\eta+5) = 4/79$. This formula has been used to obtain the parts of the M-L curves between A and B in the figure, these points marking the change first to constant guillotine factor and then to opacity by electron scattering.

Some justification of the procedure adopted in the numerical work is provided at this stage, for it is found that for values of M for which the two sources of opacity are comparable, as given by (30), the luminosity is practically the same whether calculated by means of (35) or (36).

(iii) *Variable Guillotine Factor.*—The luminosities of stars of mass less than $5\odot$ are derived from (11′) which gives a proportionality

$$L \propto (\beta\mu)^{5{\cdot}5+8{\cdot}5\zeta_1}M^{4+5\zeta_1} \tag{37}$$

where here $\zeta_1 = 5/(4\eta+7) = 5/76$. The zero point of this relation for any given μ can be determined at $M = 5\odot$ by means of the curves derived from (36).

An important feature of the calculations concerns the behaviour of the product of the parameters appearing in (35) and (36). Although the individual parameters change somewhat over the range of β involved and in the case of $\bar{m}/M$ and $\bar{x}$ particularly are raised to high powers, their combined product in L changes but little. Thus in (35) the value of $(\bar{m}/M)^3\bar{x}^{-8}$ decreases from about ·045 at $\beta = {\cdot}95$ to about ·031 at $\beta = {\cdot}25$, which for $\mu \sim 0{\cdot}5$ corresponds to a range of M from about $10\odot$ to $1000\odot$.

In the case of (36) the factor $\delta^{-2-\zeta}(\bar{m}/M)^{5+3\zeta}\bar{x}^{-14-10\zeta}$ is also subject to only slight change. Thus at $\beta = 1$ its value is about ·020 while at $\beta = 0{\cdot}4$ it is about ·031.

When the opacity is given by (15) the limiting value of the effective index is 3·83 instead of 3·25 and further integrations would be necessary to enable δ and $\bar{\chi}$ to be evaluated. Assuming however that the product of the parameters appearing in L will also be approximately constant in this case, which in view of the much smaller range of β occurring for $M < 5\odot$ is clearly a valid assumption, the simple form (37) may be used to obtain the lower portions of the mass-luminosity curves.

It is seen from the curves that the range of luminosity according to the present theory for a given mass of varying composition

$$L(M, \mu = 2) - L(M, \mu = \tfrac{1}{2})$$

has a value of about 7·5 magnitudes at $M = \odot$, but decreases steadily with increasing M, the value at $10\odot$ being about 4 magnitudes and at $500\odot$, or greater, only 2 magnitudes.

Summary

General relations for the luminosity and radius of main-series stars of given mass and composition are derived on the basis of the formula $E\rho^2T^\eta$ for the rate of generation of energy, which closely approximates to the formula obtained from physical theory. Two cases are shewn to arise. The first refers to stars in which the chief source of opacity is due to photoelectric absorption and the second to stars in which the chief source is due to electron scattering. The relations corresponding to the former case are subject to minor changes for stars of small mass owing to variation of the guillotine factor.

The determination of the parameters involved in these relations leads to the consideration of the equilibrium of the star. It is shewn that the whole star cannot be in radiative equilibrium and that the central regions must be in convective equilibrium. It is found that practically all the energy is generated within the convective core in a region containing about one-sixth of the whole mass but occupying less than 0·5 per cent. of the volume of the star.

The effect of radiation pressure is investigated on the usual assumption that it bears

a constant ratio to the total pressure. The theory leads to a more general form of the quartic equation determining the proportion of radiation pressure. The relations for luminosity and radius are capable of being put in simple form for the purposes of calculation and lead to mass-luminosity curves in good agreement with observation. In particular the average discrepancy of the Trumpler stars is only about 1 magnitude on the basis that these stars are constituted entirely of hydrogen. For stars of given mass the dispersion in luminosity for different values of μ is found to decrease with increasing mass from about 7·5 magnitudes at $M = \odot$ to about 2 magnitudes at $M = 500\odot$.

ON THE LUNAR EQUATION

Harold Jeffreys, F.R.S.

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1. The redetermination of the mass of the Moon made by Dr. H. Spencer Jones * from the Eros observations of 1930–31 involves a number of difficult statistical problems, one of which has never been satisfactorily treated. The larger part of the variation of the residuals of the asteroid's position is regular and can be attributed to small errors in the elements of the orbit adopted in the ephemeris. Superposed on this is an approximately periodic disturbance due to the Earth's monthly displacement, or rather to the correction to the trial value of the monthly displacement, which corresponds to the adopted mass of the Moon. The determination of this correction is the main problem. If this was all, the solution would be straightforward; we need only introduce corrections to the orbital elements as additional unknowns and carry out a least squares solution in the usual way. Unfortunately there is also a fluctuation depending on errors in the adopted positions of the comparison stars, which are correlated over intervals of several days. No mathematical form is suggested for this variation, but it is comparable with the correction to the monthly term, and the accuracy of the estimate of the latter will depend on our degree of success in allowing for it.

In the first place we require estimates of the first and third parts of the variation; these will be referred to together as the "fluctuation." The important point is that they are to be subtracted from the observed values to leave estimates of the lunar inequality. Consequently we must arrange our work so that they will be unaffected by the lunar inequality, and our first step is to find the fluctuation at the times when the latter is zero. Spencer Jones gives estimates approximately but not exactly at these times. I proceeded by fitting linear forms to the observed residuals ($O - T$ in Spencer Jones's paper) near the zeros of the lunar inequality; 5 to 7 values were used in each solution. The results will have some error on account of the second derivatives of the fluctuation, but this is presumably small and independent of the lunar inequality. The latter is approximately a trigonometric function, and its second derivative will be negligible near its zeros.

2. The solutions for R.A. are as follows:—

Date	Fluctuation	Weight	S.E.	Date	Fluctuation	Weight	S.E.
	″		″		″		″
1930 Oct. 13·846	+3·261	83	0·071	1931 Feb. 4·156	−0·595	1520	0·016
27·653	+3·152	111	·057	17·336	−0·947	1053	·020
Nov. 11·916	+2·847	154	·051	Mar. 3·007	−0·957	1012	·020
26·158	+2·270	259	·039	16·099	−0·949	1015	·020
Dec. 10·831	+1·732	260	·039	30·452	−0·940	566	·027
25·130	+1·198	311	·036	Apr. 12·843	−0·718	310	·036
1931 Jan. 7·986	+0·674	711	·023	27·433	−0·503	306	·036
21·595	−0·134	1100	·019				

The residuals in the course of the solution indicated a standard error for unit weight of 0″·62 based on 59 degrees of freedom.

These values were interpolated to the observed values of the time. The third difference is not very important because the intervals are nearly equal, and with a quadratic formula analogous to Bessel's, using a mean second divided difference, the contribution from the third difference is small and is antisymmetrical about the middle of

* *Memoirs R.A.S.*, 66, 11–66, 1942.

the range, where the lunar inequality is greatest.* Accordingly I used this formula twice, first to interpolate to the middle of the range and then, using these interpolates, to fill up for the intermediate days.

This method has some resemblance to the graphical one used by Dr. Spencer Jones at a later stage of his work. Mr. Melotte, on his instructions, interpolated graphically, using French curves to connect four consecutive values. If a flexible ruler had been used instead, the fourth derivative of the interpolate would be zero. Then if the join of the two intermediate values was traced the method would be equivalent (apart from errors of drawing and reading) to numerical interpolation with centred third differences.

Dr. Spencer Jones's procedure in his analytical solution A is to fit quadratics in the time to each group of three zeros of the lunar inequality; in solution B he fits cubics approximately to each set of five zeros. The polynomials are then computed and subtracted; in solution B the two end half-periods were not used. The correction to the lunar inequality is taken as $0{\cdot}1dR_a$, where $-R_a$ is the trial value of the lunar inequality. A least squares solution for d was then made from the residuals in each period after this treatment. An uncertainty was estimated by comparing the separate values for d. In solutions C and D an allowance was made for irregularities of the fluctuation and the weights were adjusted accordingly.

The methods all involve considerable modifications of the simple least squares solution and corresponding increases of the estimated uncertainty. The need for such modification may be shown simply by attempting to smooth the fluctuation by a method used occasionally in seismology.† If we have five consecutive equally spaced values y, and we fit a cubic by least squares, the corrections required are z_{-2}, z_{-1} . . . given by

$$w_{-2}z_{-2}=w_2z_2=-\lambda; \qquad w_{-1}z_{-1}=w_1z_1=4\lambda; \qquad w_0z_0=-6\lambda;$$

$$\lambda\left(\frac{1}{w_{-2}}+\frac{16}{w_{-1}}+\frac{36}{w_0}+\frac{16}{w_1}+\frac{1}{w_2}\right)=\delta^4y_0.$$

λ is computed for each value with a fourth difference opposite to it and the corresponding z_0 is applied. The two values at each end are corrected by amounts found from the nearest fourth difference. This method gives a considerable smoothing without much difficulty; it reduces irregular higher differences greatly and thereby makes interpolation easier, but it introduces a considerable positive correlation between consecutive errors, and the errors of the tabulated values are not reduced in anything like the same ratio. For this reason I adopted another method ‡ in smoothing the final seismological tables; but this method is useful in testing whether data can be smoothed without loss of accuracy. Taking my interpolates at 14-day intervals I find the following adjustments:—

Date	Interpolate	z	Date	Interpolate	z
1930 Oct. 14	+3·261	−0·012	1931 Feb. 3	−0·565	−0·022
28	3·148	+0·035	17	−0·936	+0·048
Nov. 11	2·872	−0·038	Mar. 3	−0·957	−0·026
25	2·321	+0·024	17	−0·952	−0·018
Dec. 9	1·798	−0·002	31	−0·933	+0·049
23	1·277	+0·025	Apr. 14	−0·655	−0·059
1931 Jan. 6	+0·758	−0·058	28	−0·499	+0·015
20	−0·040	+0·053			

Compared with the standard errors of the values at the zeros these suggested corrections give a χ^2 of 32·0, which is overwhelming. Further, the adjustment by this method is always incomplete and the χ^2 to be expected from a comparison made in this way is

* Jeffreys, *Journ. Lond. Math. Soc.*, 11, 270–272, 1936.

† *Bur. Centr. Séism., Trav. Sci.*, 14, 12, 1936.

‡ Jeffreys, *Theory of Probability*, pp. 171–175, 1939. A worked-out example where the data are frequencies instead of measures is given in *Ann. Eugen.*, 11, 108–114, 1941; the method may have applications in astronomy.

only about half the number of data used instead of about the number of data as in the ordinary case.* Now Spencer Jones's methods B and D make corrections at the zeros equivalent to what this method of smoothing would give if the weights were taken equal, and would give a larger χ^2 against the actual weights. The chief point of the comparison, however, is that if only errors of the orbit and of the lunar inequality were involved, the rest of the variation being random, this process would have given a χ^2 of about 8. The result shows that there must also be a correlation between consecutive errors over ranges of about 7 days. Spencer Jones is therefore right in abandoning the attempt to make a fit over the whole range by least squares and using short intervals instead; but I do not think that he has gone far enough. His monthly intervals overlap, each covering even a few days of the next but one, and his estimate of uncertainty must be regarded as based on only 6 or even 5 degrees of freedom. With so small a number a 50 per cent. over- or under-estimate of the standard error would not be very surprising, being about the 5 per cent. point.†

An obvious way of increasing the effective number of degrees of freedom is to make estimates of d for separate fortnights instead of separate months. As a rule the use of complete periods has the advantage that if we use only the half-periods there will be correlations between the higher odd harmonics and the main term, but these vanish for the complete period. This does not arise here because R_a is a definitely calculable function and there is no question of higher harmonics. It has also been suggested that if there is a general trend in the residuals the use of the complete period will remove it. But this is true only if the harmonic terms are symmetrical about the middle of the interval chosen. In this case the centres of the intervals would have to be at extreme values of R_a. This choice is not made by Spencer Jones, and in any case has a considerable compensating drawback, as we shall soon see.

There is, however, a real difficulty about the use of half-periods when the standard of comparison is a fluctuation containing a random part. To simplify the matter a little, we may regard the data as replaced by mean residuals over quarter-periods centred on the zeros and extremes of R_a. The most favourable case would be where the variation consists wholly of a systematic part (errors of orbital elements) and a random part (errors of consecutive days uncorrelated). Actually there is a correlation between consecutive days, and this will increase the mean residuals over several days. It will show itself at its worst if it is appreciable for a week; for if it lasts a shorter time it will be more like a random error, and if longer more of it will have been eliminated with the fluctuation. Consequently mean residuals over a week will provide our best estimate of the additional uncertainty introduced by this correlation. Now let the error at consecutive zeros of R be ϵ_0, ϵ_2, and in the intervening week ϵ_1. Then the analysis of a fortnight from one zero to the next will give an error $\epsilon_1 - \frac{1}{2}\epsilon_0 - \frac{1}{2}\epsilon_2$ in $0{\cdot}1dr$, where r is the mean of R_a over the intermediate week. The second and third terms are the result of our having pushed ϵ_0 and ϵ_2 into the fluctuation, but they will tend to increase the scatter of the separate estimates of d. If the weights of the fortnights are equal they will introduce a correlation of $-\frac{1}{6}$ between consecutive errors (the negative sign because the error at a zero will affect the estimates for consecutive fortnights in opposite directions). Now if τ is the standard error of a weekly mean the expectation of $(\epsilon_1 - \frac{1}{2}\epsilon_0 - \frac{1}{2}\epsilon_2)^2$ is $\frac{3}{2}\tau^2$; but with uniform weights the errors at the zeros will cancel if all the intervals are combined in the usual way, and the uncertainty of the result will be correctly found by omitting them. We can therefore estimate τ by comparing the estimates of $0{\cdot}1dr$; but we should correct by the factor $(2/3)^{\frac{1}{2}}$ before finding the uncertainty of d. With a small correlation like $-\frac{1}{6}$ it will be legitimate to obtain a first approximation by treating the errors as independent and then to correct for the correlation.

* *Proc. Camb. Phil. Soc.*, **30**, 134–138, 1934. To take the extreme case where there are only 5 entries, the resulting χ^2 is on one degree of freedom and its expectation is 1.

† Fisher, *Statistical Methods for Research Workers*, Table VI.

Alternatively we might choose the fortnights from one extreme of R_a to the next; then we effectively introduce an error $\epsilon_1 - \epsilon_3$ and the correlation between consecutive errors is $+\frac{1}{2}$. This is much more serious, since a high correlation will increase the uncertainty of any estimate of the standard error. If we used monthly intervals each from one extreme to the next but one, as was mentioned above, there would be a correlation $+\frac{1}{6}$ between consecutive errors, and we are no better off than with fortnightly intervals from one zero to the next; while at the same time we sacrifice half the degrees of freedom and lose the advantage that the errors at the ends of the ranges cancel from the final result.

Accordingly it appears that the best method that is not of prohibitive mathematical difficulty is to use fortnightly intervals each from one zero of the fluctuation to the next. The method of determining an additional standard error superposed on the known random error has already been given.* The straightforward solutions for d for the separate fortnights are as follows:—

	d	χ^2		d	χ^2
Oct. 14–27	-0.232 ± 0.052	17·8	Jan. 22–Feb. 4	-0.0120 ± 0.0042	0·1
28–Nov. 11	-0.038 ± 0.040	0·4	Feb. 5–17	-0.0355 ± 0.0049	21·1
Nov. 12–26	-0.089 ± 0.025	9·3	18–Mar. 3	-0.0094 ± 0.0056	0·4
27–Dec. 10	$+0.035 \pm 0.030$	1·6	Mar. 4–15	$+0.0245 \pm 0.0085$	19·6
Dec. 11–24	-0.035 ± 0.018	1·5	16–29	-0.0185 ± 0.0092	0·4
25–Jan. 7	-0.015 ± 0.0088	0·0	30–Apr. 12	-0.0206 ± 0.0197	0·2
Jan. 8–21	$+0.0113 \pm 0.0060$	16·7	Apr. 13–27	-0.0599 ± 0.0264	3·2
					92·3

The general weighted mean is $d = -0.0130 \pm 0.0022$. The daily residuals after allowing for this give a standard error for unit weight of 0″·624, in good agreement with that found in estimating the fluctuation near the zeros of R_a. The stated uncertainties are based on this. But χ^2 is overwhelming even when we multiply it by 2/3 to allow for its inflation due to the errors at the zeros. There is therefore clear evidence that an additional uncertainty must be taken into account, and it may be more important than the part already taken into account.

The next approximation is obtained by first multiplying each residual of d and its standard error by the extreme value of R_a in the range; we thus obtain a set of estimates nearly proportional to the mean fluctuation in the central weeks. The sum of their squares was 0·0869; but 0·0569 of this came from the two end intervals. These are under suspicion because it was not possible to use a mean second difference in interpolating the fluctuation for them, and a considerable error might arise through using the third difference to interpolate to the centre of the interval. It was therefore possible that the large residuals represented chiefly an abnormal error of interpolation, and it seemed best to omit these entries. Then the sum of squares was 0·0300; we multiply by 2/3 and subtract 0·0123, the sum of squares to be expected from the known uncertainties. Then we have 11 degrees of freedom left, and the equation for τ is

$$11\tau^2 = 0.0200 - 0.0123 = 0.0077; \qquad \tau = 0.026.$$

Then an uncertainty $\tau/R_{a(\max)}$ is combined with the uncertainties given and the items are reweighted. The result is

$$d = -0.0118 \pm 0.0040.$$

If we apply the same treatment to the end ranges and retain them at half-weight we get

$$d = -0.0127 \pm 0.0040.$$

In either case the change from the first approximation to d is trivial and we need not make a further approximation; but the change of the standard error is considerable.

* *Theory of Probability*, pp. 242–244.

It is important to have some idea of the effective number of degrees of freedom used in estimating the standard error, since the probability of a large error of the estimated d is very sensitive to it. In the simplest case this probability follows "Student's" t distribution.* Incidentally the prevalent astronomical practice of multiplying the estimated standard error by 0·6745 and calling the result the "probable error" is valid in the usual interpretation of the latter only as a limit when the number of degrees is very large and is seriously wrong when it is small. Full tables of the t distribution exist, and for convenience in their use and also in the use of the important function χ^2 it is very desirable that the so-called "probable error" should be abandoned.

In this case we may regard the standard error of d, 0·0040, as made up of 0·0022 based on about 170 degrees of freedom and 0·0033 based on 11. The departure of the t rule from the normal law is an expression of the uncertainty of the standard error; if then n' is the effective number of degrees of freedom we should have

$$\frac{4\cdot0^2}{n'}=\frac{2\cdot2^2}{170}+\frac{3\cdot3^2}{11} \qquad \text{or} \qquad \frac{16\cdot0}{n'}=\frac{4\cdot8}{170}+\frac{11\cdot2}{11},$$

which gives $n'=15\cdot3$. This estimate of n' is quite rough because strictly the probability of a compound error of this kind will not follow a simple t rule, but except for very large departures it will not be far wrong to use the t table as for $n=15$.

The value adopted by Dr. Spencer Jones from the R.A. data is

$$d=-0\cdot0137\pm0\cdot0027 \text{ (p.e.)},$$

which is not far from my solution.

3. The same method was applied to determine values of the fluctuation at the zeros of the lunar inequality in declination. The results are:

Date	Fluctuation	Weight	S.E.	Date	Fluctuation	Weight	S.E.
	″		″		″		″
1930 Oct. 5·0	−0·440	60	0·068	1931 Feb. 7·47	−0·144	1200	0·016
21·05	−0·990	225	·036	22·62	+0·077	900	·018
Nov. 6·74	−1·674	170	·042	Mar. 7·54	+0·351	600	·022
21·35	−2·245	340	·028	21·99	+0·263	900	·263
Dec. 7·22	−2·248	200	·039	Apr. 4·04	−0·057	400	·027
21·71	−2·124	120	·048	18·19	−0·236	170	·042
1931 Jan. 6·61	−1·900	530	·024	May 1·64	−0·381	220	·036
23·43	−0·832	1400	·014				

The residuals indicated a standard error for unit weight of 0″·547 based on 64 degrees of freedom.

The interpolation was carried out in the same way as for the R.A. data. The largest contribution from the second difference in the second stage of the interpolation was 0″·015, in the first 0″·064. The largest in R.A. was 0″·031. Interpolates at about 7-day intervals are as follows:—

Date		Date		Date		Date	
	″		″		″		″
1930 Oct. 5	−0·440	1930 Dec. 7	−2·456	1931 Jan. 31	−0·442	1931 Mar. 22	+0·263
13	−0·694	14	−2·324	Feb. 7	−0·161	29	+0·097
21	−0·989	22	−2·123	15	−0·011	Apr. 4	−0·058
29	−1·309	30	−2·052	23	+0·086	11	−0·159
Nov. 7	−1·682	1931 Jan. 7	−1·891	Mar. 1	+0·219	18	−0·233
14	−1·989	15	−1·406	8	+0·352	25	−0·292
21	−2·235	23	−0·865	15	+0·337	May 1	−0·342
29	−2·412						

* *Theory of Probability*, p. 118; tables in Fisher, *Statistical Methods for Research Workers*, and in Fisher and Yates, *Statistical Tables for Biological, Agricultural and Medical Research.*

The solutions for d, the coefficient of $0{\cdot}1R_\delta$, are as follows:—

	d		d
Oct. 5–20	$-0{\cdot}232 \pm 0{\cdot}179$	Jan. 24–Feb. 6	$+0{\cdot}0271 \pm 0{\cdot}0086$
21–Nov. 6	$-0{\cdot}096 \pm 0{\cdot}093$	Feb. 7–22	$-0{\cdot}0044 \pm 0{\cdot}0077$
Nov. 7–21	$-0{\cdot}026 \pm 0{\cdot}055$	23–Mar. 7	$-0{\cdot}0074 \pm 0{\cdot}0101$
22–Dec. 6	$-0{\cdot}101 \pm 0{\cdot}042$	Mar. 8–21	$-0{\cdot}0087 \pm 0{\cdot}0097$
Dec. 8–21	$-0{\cdot}205 \pm 0{\cdot}029$	22–Apr. 4	$-0{\cdot}074 \pm 0{\cdot}0160$
22–Jan. 6	$+0{\cdot}0165 \pm 0{\cdot}0153$	Apr. 5–18	$-0{\cdot}116 \pm 0{\cdot}0281$
Jan. 7–23	$-0{\cdot}0910 \pm 0{\cdot}0115$	20–May 1	$-0{\cdot}165 \pm 0{\cdot}041$

The general mean is $d = -0{\cdot}0184$, and the standard error for unit weight $0''{\cdot}876$. With this standard error that of d is 0·0051. Unlike what was found for the right ascensions, the standard error for unit weight found from the whole of the data is decidedly greater than was found in making the determinations for the times of zero lunar inequality, $0''{\cdot}547$. Actually in most of the separate fortnights the residuals would be consistent with the latter estimate. If we allow for the general value of d the exceptions are for the fortnights December 8–21, January 7–23 and January 24 to February 6, for which the standard errors for unit weight are found to be $1''{\cdot}16$, $1''{\cdot}35$ and $1''{\cdot}00$ respectively. All are associated with highly anomalous estimates of d, and the first two would become normal if we allowed for their own estimates of d instead of the summary one. It appears, therefore, that the lower standard error for unit weight is the more representative of the random error; the other is inflated by the inclusion of specially large contributions from the anomalies that persisted for a week. On the other hand the lower value is a little less than the corresponding values from the right ascensions, though not significantly so. The best procedure is probably to use a random standard error for unit weight of $0''{\cdot}60$ or so and attribute the rest to the τ variation. The uncertainties given under the separate values of d correspond to $0''{\cdot}60$ for unit weight; the corresponding one for the general mean would be 0·0038. But the discrepancies are so large that it is not necessary to compute χ^2 to reveal them.

We proceed to estimate τ as for the R.A. data; the first approximation is $\tau = 0{\cdot}066$; and the revised weighted mean is $-0{\cdot}062 \pm 0{\cdot}020$. The change is 2·2 times the new standard error and we have a clear case of what Spencer Jones calls the instability of the solution from the declinations. On inspection of the data the reason is easily seen. There are four fortnights from January 24 to March 21 with unmodified standard errors of 0·0101 or less, and all give values of d with positive residuals against the unmodified mean. Of the rest, all give negative residuals except one. The result of any uniform additional standard error is therefore necessarily to make the mean smaller. The same is shown by Spencer Jones's solutions C and D, which include such an uncertainty and give smaller means than his A and B, which do not include it, but redetermine the standard error from the discordances as if it was wholly random. The change would have been smaller if the random standard error had been taken as $0''{\cdot}876$, but there seem to me to be adequate reasons against this.

The revised solution is also seriously different from the determination from the right ascensions, for which there was no corresponding difficulty. It seems that the introduction of an uncertainty affecting every fortnight equally is doubtfully valid. Some suspicion attaches to it for another reason. Seven of the apparent uncertainties of the separate estimates of d are 0·0160 or less; three of these agree within 0·0043. This would not be expected if there was a general uncertainty that would affect these values by $\tau/(R_{\delta\,\max})$, which would be about 0·03 for them. Similarly for the right ascensions there are seven with apparent standard errors under 0·0100 and four of these agree within 0·009, which again would not be expected if an additional uncertainty of at least 0·009 was to be included, particularly since the tendency of the method is to increase the differences between consecutive entries. In fact the number that are consistent with

the same value, within half the apparent standard errors, is in accordance with the supposition that these standard errors are genuine. If this is so, it would be valid to accept them at their face value and combine the values of d accordingly. This procedure looks questionable at first sight, because the full probability solution is known for the case where all the uncertainties are equal and both the random and the additional variation follow the normal law, and the correct uncertainty would be given by the τ method in any case.* But the question is whether the additional variation does follow the normal law. The concentration at small residuals and the great range of the large ones suggest that it is more like a Pearson Type VII law with a low index. For such a law the second moment is hopeless as an estimate of the spread, and the method will be wrong. We cannot, however, expect to get a good determination of the index from about 14 values, several of which have a large uncertainty with any treatment. What this would imply is that the accuracy of the positions of the comparison stars is patchy; the determinations of the mean positions have very different weights. A full analysis is therefore beyond the present resources of probability theory. Nevertheless a rough solution can be made, embodying the main principle that large deviations in relation to the apparent standard error must be attributed mainly to errors of the standard stars and therefore receive low weight in estimating the lunar inequality, while small ones can be used at nearly full weight. We take a trial value of d, and divide each residual by the uncorrected standard error; then we multiply the weight corresponding to the latter by the factor used in fitting a Type VII law of index 4 to give a revised weight.† A new weighted mean is formed and the process repeated. The unsatisfactory feature of the method is that the errors of the star places have nothing to do with the number of plates showing Eros and those stars, and it is mainly due to the latter that the uncorrected weights vary. Also the particular choice of the index was made for a quite different reason. But it does ensure that large residuals receive low weight, whether they are due to few observations or to large errors in the star places, and this will be required by any valid solution if the law of error of the star places is of the type considered; and the resulting uncertainty of the weighted mean is not very sensitive to the choice of m. This suggestion has two previous analogues. One is in the construction of seismological tables, where a method of continuous weighting was used to allow for the background of large errors superposed on the concentration of small ones. The other was in the analysis of the variation of latitude.‡ This can be regarded as a damped periodic motion with irregular disturbances. It was found possible to identify a long interval when the irregular disturbances were small, and the final results were based on this interval.

For the declinations the first approximation was taken as $d = -0{\cdot}0184$; later approximations were $-0{\cdot}0118$, $-0{\cdot}0090$, $-0{\cdot}0083$, $-0{\cdot}0082$. The standard error of the last is 0·0047. Though based on totally different principles the final result agrees very well with that given by Spencer Jones's graphical method, which is $-0{\cdot}0097 \pm 0{\cdot}0042$ (p.e.). The same method applied to the R.A. data gave $-0{\cdot}0125 \pm 0{\cdot}0027$, and the two determinations are in good agreement; their weighted mean is

$$d = -0{\cdot}0114 \pm 0{\cdot}0023 \text{ (s.e.)}.$$

This standard error is a little too low because it neglects the uncertainties of the ends of the ranges. So long as the weights are uniform or vary gradually these cancel in forming the solution, but the modification of the weights makes them irregular and it will be best to apply the correcting factor $(3/2)^{\frac{1}{2}}$ to take this into account. The final result is then

$$d = -0{\cdot}0114 \pm 0{\cdot}0028 \text{ (s.e.)}.$$

Spencer Jones gives

$$d = -0{\cdot}0133 \pm 0{\cdot}0023 \text{ (p.e.)}.$$

* *Theory of Probability*, pp. 229–234.

† *Theory of Probability*, p. 254.

‡ *M.N.*, **100**, 139–155, 1940.

I think that the former is somewhat the better, particularly as it avoids the use of a graphical method, which always arouses grave suspicion in me. The substantial difference between my first solution from the declinations and Spencer Jones's graphical solution must, I think, be due to the use of French (railway) curves; for with interpolation by means of a flexible ruler the results should agree. Also the uncertainty, though its determination is not altogether satisfactory, depends on a method of weighting that seems to correspond better with the distribution of the separate estimates. The difference between Spencer Jones's result and mine is, however, quite small.

The corresponding value of the coefficient of the lunar inequality is

$$L = 6''{\cdot}4305(1 - 0{\cdot}1d) = 6''{\cdot}4378 \pm 0{\cdot}0018 \text{ (s.e.)}.$$

The "probable error" would be $0''{\cdot}0012$. If we combine Spencer Jones's determination with the three earlier ones quoted by him we get $L = 6''{\cdot}4371$, with $\chi^2 = 6{\cdot}9$ on 3 degrees of freedom. Combining mine with the earlier ones similarly we get $L = 6''{\cdot}4367 \pm 0''{\cdot}0017$, $\chi^2 = 6{\cdot}2$. Both values of χ^2 are rather large, $P = 0{\cdot}07$ and $P = 0{\cdot}10$ respectively, but not enough to say that the data have a systematic difference. I should, however, be inclined to suspect that some of the earlier estimates of uncertainty are a little too small.

5. With regard to the last part of Spencer Jones's paper, in which he makes a general adjustment of the related astronomical constants, I think that one set of equations rests on an incorrect geophysical hypothesis and should be rejected. De Sitter, using a second order theory of the ellipticity of the Earth based on Radau's approximation, found that, subject to the hypothesis that the Earth's interior is in a state of pure hydrostatic stress, the ellipticity and the two related constants, $\mathcal{J} = \frac{3}{2}(C - A)/Eb^2$ and β, the coefficient of $-\sin^2\phi$ in gravity, could be calculated from the precessional constant with a probable error of $0{\cdot}00000011$, corresponding to $0{\cdot}10$ in the reciprocal of the ellipticity.* Since the upper layers at any rate are not in a hydrostatic state, de Sitter made another solution assuming the distribution of density with depth in the upper layers assumed in isostatic theory, with hydrostatic conditions below, and found that the change in the calculated ellipticity was inappreciable. He also made a solution treating the surface inequalities as added loads, and estimates † that there would be a change of about two units in the denominator of the computed ellipticity. But the widespread gravity anomalies actually found are not noticeably correlated with surface elevation, and it happens that the elliptic harmonic in the surface elevation is much smaller than some of those of the third degree. This test is therefore no check at all. The analysis of gravity observations and pendulum deflexions has shown considerable departures corresponding to low harmonics. We can estimate the stresses in the upper layers needed to support the additional masses that these indicate, assuming the lower layers hydrostatic, but they are found to require a totally impossible strength.‡ We can get consistency only by assuming some stress-differences in the lower layer. Heiskanen's ellipticity of the equator is an attempt to represent two of these low harmonics. It was premature in the sense that his data would not separate the possible spherical harmonics in observed gravity, on account of extreme non-orthogonality of the normal equations, but it was enough to show that notable departures from the predictions of the hydrostatic theory exist. In my analysis I found longitude terms in different latitudes with coefficients up to 16 milligals.§ If there are such departures from hydrostatic theory we cannot expect its prediction of the variation of gravity from pole to equator to be right within a standard error of about 8 milligals, corresponding to one of 8×10^{-6} in the ellipticity, and 7 times de Sitter's uncertainty. It might even be held that since the rotation of the Earth has probably changed, the ellipticity term is more likely than the

* *B.A.N.*, 55, 97–108, 1924; 129, 57–61, 1927.
† *B.A.N.*, 55, 102, 1924.
‡ Jeffreys, *Geophys. Suppl.*, 3, 69, 1932.
§ *Geophys. Suppl.*, 5, 16, 1941.

others to show an anomaly; and the corresponding data for the Moon are quite inconsistent with hydrostatic theory. Spencer Jones gives 0·054 as the p.e. of the reciprocal of the ellipticity, which is still lower than de Sitter's, mainly owing to the reduction of the uncertainty of the solar parallax and the mass of the Moon. I think that all parts of the theory that depend on the hydrostatic relation should be rejected so that, as in other problems relating to the distribution of gravity, we may be able to see how far it is wrong. De Sitter's equations, however, involve it in such intricate ways that its removal would be a formidable task.

In view of the importance attached to the discrepancy relating to the constant of nutation, however, it is worth while to make a rough adaptation of Spencer Jones's solution to see whether the abandonment of the relation would reduce this discrepancy appreciably. His least squares solution, compared with his equations (1′) to (7′), gives $\chi^2 = 7{\cdot}8$, of which 4·5 comes from (4′), the nutation equation, and 1·6 from (7′), the lunar parallax. This is hardly evidence of a discrepancy, since there are 4 degrees of freedom and P, the chance of a larger χ^2 arising by chance, is 0·10. To make the modification that we need, we notice that in (3′) and (4′), the precession and nutation equations, it is really the dynamical ellipticity that enters; in (1′), (5′), (6′) and (7′) it is the ellipticity of the surface. We therefore replace w in these equations by θ and treat θ as a new unknown. If w is then replaced by θ in the de Sitter equation giving the surface ellipticity in terms of the dynamical ellipticity we should get a valid estimate of the former. The solution is

	H. J.	H. S. J.	H. S. J. – H. J.
$x =$	$-0{\cdot}001456 \pm 0{\cdot}000169$	$-0{\cdot}001437 \pm 0{\cdot}000168$	$+{\cdot}000019$
$z =$	$-0{\cdot}003095 \pm 0{\cdot}000381$	$-0{\cdot}002967 \pm 0{\cdot}000377$	$+{\cdot}000128$
$w' =$	$-0{\cdot}000007 \pm 0{\cdot}000047$	$-0{\cdot}000005 \pm 0{\cdot}000047$	$+{\cdot}000002$
$\theta =$	$+0{\cdot}00023 \pm 0{\cdot}00161$	$-0{\cdot}002007 \pm 0{\cdot}000254$	$-{\cdot}00224$
$w =$	$-0{\cdot}002095 \pm 0{\cdot}000262$	$-0{\cdot}002007 \pm 0{\cdot}000254$	$+{\cdot}000088$

I have converted his probable errors into standard errors. The change in θ due to not identifying it with w is 9 times his standard error and 1·4 times mine. Comparing with the original equations we have

		Calc.	Obs.	O – C	χ^2
Lunar inequality	(1′)	$+0{\cdot}001614$	$+0{\cdot}001671 \pm 0{\cdot}000345$	$+0{\cdot}000057$	0·03
Solar parallax	(2′)	$-0{\cdot}001456$	$-0{\cdot}001477 \pm 0{\cdot}000169$	$-0{\cdot}000021$	0·02
Precession	(3′)	$-0{\cdot}000007$	$0{\cdot}000000 \pm 0{\cdot}000047$	$+0{\cdot}000007$	0·02
Nutation	(4′)	$+0{\cdot}000962$	$-0{\cdot}000510 \pm 0{\cdot}000676$	$-0{\cdot}001472$	4·75
$\mathcal{J}$ from perigee, etc.	(5′)	$+0{\cdot}000382$	$+0{\cdot}000153 \pm 0{\cdot}003268$	$-0{\cdot}000229$	0·00
Gravity	(6′)	$-0{\cdot}000119$	$+0{\cdot}000237 \pm 0{\cdot}001546$	$+0{\cdot}000356$	0·05
Lunar parallax	(7′)	$-0{\cdot}001321$	$-0{\cdot}003890 \pm 0{\cdot}002624$	$-0{\cdot}002569$	0·96
					5·83

Since four unknowns have been determined the new χ^2 is on three degrees of freedom, and $P = 0{\cdot}18$. By all ordinary standards of comparison, therefore, the general agreement is quite good. If we choose to consider the nutation separately as deviating by nearly 2·2 times its standard error we might suspect it if it stood alone; but it is one equation selected out of seven and the reason for regarding the difference as anything but a random error is diminished accordingly. The selection is really more severe because an unusually small χ^2 was also found in the equations combined to produce (5′).

The value used for the nutation is that of Spencer Jones, $9''{\cdot}2134 \pm 0{\cdot}0062$ (s.e.). Other values such as Brouwer's $9''{\cdot}2075 \pm 0''{\cdot}0030$ (s.e.) would give a much more serious discordance. My own impression is that the method used by Spencer Jones is valid and his uncertainty genuine *, adequate tests for independence of the errors having been applied. It may be remarked that the removal of the restriction $\theta = w$ has hardly affected

* *M.N.*, 99, 211–216, 1939.

the contribution to χ^2 from the nutation equation, and the discrepancy at this point cannot be attributed to the hydrostatic hypothesis.

The increased standard error of θ indicates to what an extent Spencer Jones's depends on de Sitter's equation. I have not carried the matter further at present, because it seems to me that we have reached a stage where full consistency requires a large number of small adjustments, and it would be best to make them together. The first is certainly to make the changes in de Sitter's theory required to remove the hydrostatic hypothesis and use the relations between $\mathcal{J}$, e and β that depend wholly on the theory of the external field. What has been done above is an attempt to remove complications that should never have been introduced. I think that the value of the lunar inequality found here is slightly better than Spencer Jones's, though the difference is not important. The absolute value of gravity also needs a check. The present situation is that two determinations have been made at the National Physical Laboratory and the Bureau of Standards, and their consistency has been checked by Bullard and Browne by comparing the periods of six pendulums transported between the two sites. But gravity in most places depends on the comparison of periods of pendulums with those of the same pendulums at Potsdam, and consequently we do not know the mean value of gravity until we know the correction to be applied to the old absolute value at Potsdam. The Bureau of Standards has been compared with Potsdam, but the connection of Teddington with Potsdam is weak *, resting on an unchecked comparison with Greenwich. It would therefore be premature to adopt a definite value for the correction, since its accuracy would be much improved by a direct comparison of Teddington with Potsdam, or even with Cambridge or Greenwich.

6. Dr. Spencer Jones points out a numerical mistake in my solution for $\mathcal{J}$ and related constants †: in my equation (2′) $+0{\cdot}009$ should be $-0{\cdot}023$. Correcting this I get

$$\begin{aligned} \mathcal{J} &= 0{\cdot}0016390 \pm 0{\cdot}0000062, \\ 1/\epsilon &= 296{\cdot}94 \pm 0{\cdot}56, \\ \epsilon &= 0{\cdot}0033677 \pm 0{\cdot}0000062, \end{aligned}$$

and the coefficient of $g_0 \sin^2 \phi$ in gravity is $0{\cdot}0052882 \pm 0{\cdot}0000062$. For the Moon,

$$\begin{aligned} \mathcal{J}' + \tfrac{1}{2}K' &= 0{\cdot}0003733 \pm 0{\cdot}0000016, \\ K' &= 0{\cdot}000075 \pm 0{\cdot}000034. \end{aligned}$$

The uncertainties are standard errors. χ^2 is increased from 0·42 to 0·59 on 3 degrees of freedom and is still very small.

I must express my thanks to Dr. Spencer Jones for his assistance over many questions that have arisen in the course of this paper.

Summary

The data used by Dr. H. Spencer Jones in his redetermination of the mass of the Moon have been rediscussed by two methods, with the object of obtaining an estimate of uncertainty based on a larger number of degrees of freedom. Fortnightly intervals were used after subtracting an estimate of the part of the variation of the residuals not attributable to the Moon. In the first place an additional uncertainty of the mean residual over a week, following the normal law, was used, but it led to a serious discrepancy between the estimates of d from the right ascensions and declinations. The fortnightly residuals themselves did not appear consistent with the normal law of error, and another solution

* E. C. Bullard and H. L. P. Jolly, *Geophys. Suppl.*, **3**, 449, 1936.

† *M.N.*, **101**, 34–36, 1941.

was made with weights depending on the ratio of each residual to its apparent standard error. This gave

From R.A.,	$d = -0{\cdot}0125 \pm 0{\cdot}0033$.
From Dec.,	$d = -0{\cdot}0082 \pm 0{\cdot}0058$.
Total	$d = -0{\cdot}0114 \pm 0{\cdot}0028$ (s.e.).
	$L = 6''{\cdot}4378 \pm 0{\cdot}0018$.

The assumption of hydrostatic equilibrium in the interior of the Earth, used by de Sitter to determine the ellipticity from $(C-A)/C$, is criticized. The data used by Spencer Jones are recombined without this assumption. It does not appear that the value of the nutation coefficient obtained by him is in serious disagreement with the other relevant data, though the smaller values given by other writers would be.

ON THE STRUCTURE OF GALACTIC CLUSTERS

George Alter

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1. *Introduction.*—The structure of a galactic cluster depends on the cluster's development, which is of two kinds: the dynamical evolution of the whole unit and the spectral evolution of its members. Though the internal dynamical forces are not fully known, the influence of the external ones is fairly well established. In this paper a spectral evolution, based on Trumpler's classification, is suggested and a statistical relation found between the dynamical and spectral constitutions of clusters. This relation indicates a development analogous to the Russell diagram.

2. *Dynamical Structure.*—It is known that the galactic clusters disintegrate under the gravitational influence of the galactic nucleus and of intruding stars. This results in the loss of stars and changes of diameters. The data on dynamical structure are, however, very scarce. Only few catalogues of proper motions have been published and the radial velocities by Hayford * do not include, for any cluster, a sufficient number of stars to determine the residuals due to the positions of the stars within the cluster. The orientations of elliptical clusters, suggested by Mineur † as indicating their dynamical states, are not reliable because of the uncertainty of estimating cluster outlines. A comparison of the orientations of 38 clusters given by Shapley ‡ with those of Collinder § shows that, within a range of 0° to 90°, the mean scatter is $\pm 27°$, and Shapley's description (p. 97) of orientations in Cassiopeia and Sagittarius regions is contradictory to that of Collinder (p. A53).

Our knowledge of the dynamical structure of clusters is thus very restricted and may be expressed by the general statement that they are stellar congregations showing a closer gravitational relation than an average galactic star field of the same star density, and representing a compound unit in the galactic gravitational field.

3. *Spectral Constitution.*—Trumpler || proposed a hypothetical evolution of clusters, based on a detailed classification according to two parameters: the presence of yellow or red giant stars and the earliest star type, within the cluster.

Trumpler's classification can be simplified by regrouping into four classes. His types may first be divided into (*a*) clusters containing yellow or red giants in addition to main sequence stars ("giant branch"), and (*b*) those containing only main sequence stars, including his "doubtful" group 1–2 ("main branch"). Each of these may then be subdivided in two temperature types according to the hottest type present—a "cool type" with the hottest stars of types A–F, and a "hot type" containing also B and/or O stars. The four cluster groups will be called giant cool (GC), giant hot (GH), main hot (MH) and main cool (MC) classes. Then it is suggested that clusters develop from the GC through GH and MH to the MC class, with a certain indeterminacy near the central classes where short transitions, evading one or both classes, may occur (as pointed out by Trumpler). This hypothesis is supported by Kuiper's ¶ work on hydrogen contents and by Trumpler's own data.

The material used here is based on 100 clusters of Trumpler's Table 3, whose distribution in the four classes just described is 24, 6, 64, 6, respectively. The explana-

* *L.O.B.*, 16, 62, 1932. † *An. d'Ap.*, 2, 140, 1939. ‡ *Star Clusters*, 1930.
§ *Lund Annals*, 2, 1931. || *P.A.S.P.*, 37, 316, 1925; *L.O.B.*, 14, 158, 1930.
¶ *Ap. J.*, 86, 176, 1937.

tion of the small number in GH is given by Trumpler * and Kuiper †; the preponderance of MH is due to selection (high luminosity of hot types), as noticed by Trumpler; the small number of MC is a consequence of the same selection; GC is not so much affected by selection. This point is discussed more fully later.

Table I

Frequency Distribution of Clusters of Various Types

	r	*m*	*p*
GC	11	4	9
GH	5	1	0
MH	12	25	27
MC	2	1	3
Total G	16	5	9
,, M	14	26	30
O	2	5	3
B	10	20	24

The first part of Table I gives the cluster distribution among the four classes mentioned above and the three characteristics of star numbers introduced by Trumpler, *i.e.* rich (*r*), moderately rich (*m*) and poor (*p*). The second part shows that, while among the G branch there are almost twice as many rich clusters as poor, in the M branch the relation is reversed. This corresponds to what should be expected if our arrangement of classes represents cluster development: a loss of stars with increasing age. This evolution is still more pronounced in the hot classes, as will be seen in lines 2 and 3. Even the step from the clusters containing O stars to those containing only B stars (in the MH class) brings the expected change (last two lines).

The selection mentioned above can be shown to be present by a relation between cluster class and distance from the observer. Assuming a random cluster distribution, there should be no such relation if every cluster were recognized. Actually the average distance (in kiloparsecs) of the four cluster classes is 0·71, 0·85, 1·03, 0·58, respectively. The cool clusters seem to be restricted to smaller distances, as shown in Table II.

Table II

Distance (Parsecs)	0·0–0·5	0·5–1·0	1·0–1·5	1·5–2·0	2·0–2·5
C	10	17	2	1	0
H	20	22	16	9	3

A secondary selection, due to the loss of poor clusters with increasing distance, might be suspected, but no noticeable systematic change with increasing distance can be established. Thus it seems evident that the selection is only a function of star luminosity.

The results of this section depend on an assumed dynamical influence of the galaxy on

* *P.A.S.P.*, 37, 317, 1925. † *H.B.*, 903, 2, 1936.

cluster structure and we may attempt to confirm the existence of such an influence in Trumpler's data. Bok * shows that the stability of the clusters is a function of the tidal forces of the galaxy, and that the forces are larger for clusters nearer to the galactic centre. We should therefore expect these clusters to have disintegrated more than those farther out. If we draw a sphere around the galactic nucleus with the distance of the Sun as radius, 46 clusters lie inside and 54 outside. The percentage of each of the four classes is nearly the same in both regions, as will be seen from the first part of Table III, and the results will therefore be free from selection due to a different stage of development.

The stage of disintegration is indicated in the last three columns. While among the outside clusters only about a quarter is poor, half of the inside clusters show this quality, as was to be expected. This accelerating influence of the galactic position is not, however, strong enough to affect the distribution of the clusters in the four classes. Thus the cluster distance from the galactic centre is a further argument in cluster development.

TABLE III

		GC	GH	MH	MC	*r*	*m*	*p*
Inner Region	No.	12	3	29	2	10	12	24
	Percentage	26	7	63	4	22	26	52
Outer Region	No.	12	3	35	4	20	19	15
	Percentage	22	6	65	7	37	35	28

This section seems therefore to give evidence that the clusters, while slowly disintegrating, develop in a way similar to the Russell diagram, and that the disintegration proceeds quicker in clusters nearer to the galactic centre. Further and more detailed investigation of these questions will be possible either when we have advanced in the spectral determination of the fainter clusters or with the help of CI determinations, taking due regard to intervening absorption.

4. *Dimensions.*—The determination of dimensions is hampered by the difficulties of defining the boundaries of clusters. If we disregard ellipticity, for the reasons given in section 2, and restrict ourselves to the recognizable denser cluster core, disregarding outside members (which may be found up to 2 or 3 times the assumed diameter), then the volumes of clusters range from 10 to 1000 cubic parsecs, using the 100 Trumpler clusters. It is improbable that larger ones are omitted, because we can reasonably assume that the collection of known clusters represents a well-balanced average mixture of all clusters. The lower volume limit, however, may be smaller because it is possible that some clusters may have been overlooked in the many accidental small stellar condensations; also, the question of the minimum number of stars which can constitute a cluster is still unsettled.

Cluster dimensions also depend on gravitational relations, as mentioned in section 2. With increasing age the *true* cluster diameter, covering every cluster member, increases, and its value can be determined by a detailed investigation on the membership of stars surrounding the recognizable denser part. This increase will cease when the star density at the boundary becomes about equal to the star density of the general star field and the escaping stars lose their gravitational relation to the cluster.

The *estimated* diameter, however, which is derived from the apparent cluster shape or density contrast (and used in most investigations) takes a different course of development. It can only be measured when the density level is about 30 per cent. or more higher than that of the background.† As long as the star density at the cluster outline is above this

* *H.C.*, 384, 1934. † Trumpler, *Ap. J.*, 91, 198, 1940.

level the estimated diameter will increase with disintegration. As soon, however, as the outline density drops under this level the diameter will stop increasing, and the further loss of stars will cause a continuing decrease of density, with a continuous shrinking of the estimated outlines. The point where the maximum may be reached also depends on the initial star density: the diameters of rich clusters will increase much longer than those of moderately rich or poor ones. Also, it may be expected that the advanced disintegration of clusters nearer the galactic centre will be recognizable in the diameters.

Trumpler's data give the following results for estimated diameters in parsecs, with the number of clusters in brackets (the diameters were computed from distance and angular diameter, Table 16): the average diameter of all clusters is 4·98 (100); the average diameter of hot clusters 5·17 (71) and of cool clusters 4·54 (29), thus showing the expected increase in the central (hot) part of the evolution.

Table IV

	GC	GH	MH	MC
All	4·52 (24)	6·62 (6)	5·10 (64)	3·92 (6)
I.R.	4·71 (12)	6·47 (3)	4·68 (29)	3·20 (2)
O.R.	4·21 (12)	6·77 (3)	5·48 (35)	4·28 (4)
Raab	8·14 (16)	10·14 (5)	8·78 (32)	6·84 (5)
r	4·65 (11)	5·92 (5)	6·69 (12)	4·65 (2)
m	4·93 (4)	10·10 (1)	5·90 (25)	4·70 (1)
p	4·02 (9)	...	3·69 (27)	3·17 (3)

The first line of Table IV gives the average diameters of the four classes showing the expected increase followed by a decrease. The second and third lines separate the clusters according to galactic position, showing that the diameters of clusters nearer the galactic centre decrease more rapidly than those of the outer clusters. (This difference would be even larger if we assume an absorption increasing towards the galactic centre.) The fourth line gives for comparison Raab's * diameters, which were not estimated but determined by star-counting. (This method is more reliable and the results are obviously larger, by about 75 per cent.) Raab's diameter sequence shows the same behaviour as Trumpler's. The second part of the table shows that the disintegration of poor clusters is much quicker than that of rich ones, in good agreement with expectation.

It is at present not possible to apply a similar treatment to the true diameters because there is no reliable series of measurements of these values. We are, however, able to use the diagrams by C. Anger Rieke † for an indication of the behaviour of the true diameters in relation to the estimated ones. In these diagrams the apparent star magnitudes are plotted against *H* line intensities (representing spectral types), and three kinds of symbols are used to distinguish stars within the estimated diameter, from those within twice and three times this distance. The positions of these three symbols in relation to the resulting curve (representing cluster membership) indicate the values of the true diameters. Table V gives the distribution of the 38 clusters for which sufficient data were available among classes and true diameters (in units of estimated diameters). The distribution shows a tendency of advanced stages to become relatively larger, as expected.

* *Lund Medd.*, Ser. II, No. 28, 1922.
† *H.C.*, 397, 1935.

TABLE V

	GC	GH	MH	MC
1	1	0	1	0
2	0	3	9	0
3	0	2	20	2

The distribution of diameters through observed distances shows the expected selection of larger diameters in larger distances. The average distances (in kpc.) for different diameters are:

Diameters (psc.)	2–4	4–6	6–12
Av. Dist.	0·82 (42)	0·93 (40)	1·01 (18)

5. *Star Density.*—The individual numbers of cluster stars do not contribute directly to our knowledge of the general cluster qualities, but they can give us some information in their relation to the cluster volumes, as average densities. The most reliable and homogeneous source of information for this purpose is Raab's survey. This, with Trumpler's distances (Raab's not being reliable), was used in compiling Table VI, which is thus corrected for galactic absorption. (A comparison of these data with Trumpler's could not be made because Trumpler does not give individual star numbers.) The densities are given in number per cubic parsec.

TABLE VI

Star Density	Diameter (psc.)			Star Number			Branch		Gal. Pos.		Distance (kpc.)		
	0–5	5–10	> 10	*r*	*m*	*p*	G	M	I.R.	O.R.	0–0·5	0·5–1	> 1
> 0·4	9	21	0	14	10	6	12	18	7	23	13	12	5
< 0·4	0	22	12	10	15	9	9	25	18	16	5	18	11

The first two parts show the expected dependence of density on diameter and star number. While the moderately rich and poor clusters show the same density distribution, the proportion of the numbers of dense to sparse clusters among the rich ones is double that among the *m* and *p* clusters, corresponding to Trumpler's result (*L.O.B.*, **14**, Table 7), who happens to get double the density for rich clusters as for the others, although he uses his own diameters and only partly Raab's star numbers.

The table also shows that among the giant branch the larger number of clusters are dense while among the main branch the larger part is sparse, as should be expected because of the disintegration, and confirms the influence of distance from the galactic centre: the average star density increases from the inner to the outer regions.

Finally, the last part checks the selection, mentioned above, viz. the prevalence of denser clusters at the smaller distances from the observer.

The densities used above range between 2·5 and 0·05 stars per cubic parsec, but they are actually much smaller than the true ones because many cluster stars are omitted on account of their faintness, and naturally the number of these omissions increases with increasing cluster distance.* The density in the vicinity of the Sun, using Kuiper's †

* The influence of this selection also appears in Trumpler's Tables 2 to 4, *Ap. J.*, **91**, 199 1940. † *Ap. J.*, **95**, 201, 1942.

data, is 0·05. If we exclude the stars with absolute magnitude > 7 as too faint at the larger cluster distances the density drops to about 0·01. The difference between this density and the apparent lower limit of cluster density (0·05) is probably due to the fact that we do not recognize as such clusters of lower density.

6. *General Conclusions.*—The result of the preceding sections may perhaps best be summarized by an attempt at a working description.

Galactic clusters are star congregations with closer gravitational relation than that of the surrounding star field. If we regard the statistical relations between the dimensions, star numbers, distances from the galactic centre and spectral constitutions (based on Trumpler's classification) as representing an evolutionary sequence, then the clusters develop along a life-line similar to that of the Russell diagram. During this evolution the clusters are subjected to the influence of the galactic field. The initial volume of 10 to 100 cubic parsecs increases until disintegration leads to the loss of stars merging with the surrounding star field. The lowest recognized average spatial star density within a cluster is about five times that of the star density in the vicinity of the Sun, while the highest is about 250 times as much. The true values (within the estimated outline) are however higher, because the star numbers used for determination are restricted by the magnitude limit of the photographs. The largest observed volume is about 1000 cubic parsecs.

Globular clusters are distinguished from galactic ones by their larger star number, higher central density and steeper density gradient, as well as by the difference of distances and celestial distribution, and the presence of variables which are lacking in galactic clusters. The origin of galactic clusters has not yet been established, and the suggestion of being decay products of globular clusters is partly contradicted by the differences between them, mentioned above, although it may be argued that the influence of the galaxy has disintegrated all globular clusters within its boundaries to galactic ones.

The definition of the apparent structure of galactic clusters is, to a certain degree, an individual one, because of the necessarily individual estimates of the degrees of concentration and outline character. It is advisable to use a conception of outline definition which does not permit a description of clusters as being simultaneously of no appreciable concentration and no well-defined outline, because this case includes hundreds of accidental star groups, especially in the Milky Way.

Summary

The structure of galactic clusters is shown to be a function both of the gravitational influence of the galaxy and of the age in the spectral evolution. Based on Trumpler's classification, the clusters are divided into four classes representing in succession this development, which is analogous to Russell's diagram of stellar evolution.

Tables are given showing the relations between these four classes and various structural properties of galactic clusters, all of which support the suggested scheme of evolution.

An attempt is made to give a general working description, or definition, of galactic clusters.

Norman Lockyer Observatory,
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THE SYSTEM OF GALACTIC CLUSTERS IN RELATION TO THE GALAXY

George Alter

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1. *Introduction.*—A recent investigation of cluster properties * made it desirable to ascertain the position of the galactic centre from material based on clusters alone. The available determinations range from 325° to 370° galactic longitude and are almost entirely based on the differential rotational effect for various kinds of celestial objects. (A comprehensive survey of these investigations will be found in Lindblad's article on the Milky Way in *Handbuch der Astrophysik*. Later determinations also keep within the range.) It seemed therefore advisable to use the material of the author's investigation * for an attempt at such a location in order to get a homogeneous result. The material consisted of the positions of 334 galactic clusters listed by Trumpler.†

2. *Method.*—Two kinds of distribution in galactic longitude—those of average cluster distance from observer, and of cluster number—are supposed to indicate the required direction. In order to smooth the scatter due to the irregular cluster distribution, the disk-shaped cloud of clusters is cut by a plane through the position of the Sun and perpendicular to the galactic plane in two parts, A and B, and the mean distance of all clusters contained in either part represents the mean cluster distance in the longitudes vertical to the cutting plane. In turning this plane in steps of 10° along the galactic circle 36 mean cluster distances are obtained. However, by the appearance and disappearance of each 10° sector, these steps introduce a secondary scatter which is smoothed by averaging five consecutive values. A similar procedure is adopted for the cluster number.

3. *Results.*—Fig. 1 gives the resulting curves. Curve I represents the differences of the distances between the two opposite disk halves, A – B. (The average cluster distance is about 1970 psc.) Curve II gives the difference of the cluster numbers, B – A.

To account for a difference in distances we have to consider that the distances are determined by means of spectral types for 100 clusters, and for the remainder by the assumption of a relation between apparent cluster class and linear diameter. Trumpler found, however, a relation between linear diameter and distance and removed it by assuming an absorption which is, on the average, uniform. (The presence of absorption could be proved by the presence of a colour excess with different clusters.) None of the quantities used (distance, linear diameter or angular diameter) is likely to have a real relation to galactic longitude, and the observed relation can only be explained by assuming an absorption non-uniformly distributed in longitude. If we assume, for instance, an absorbing stratum with density decreasing outwards from the galactic centre then we shall find an absorption coefficient increasing towards the galactic centre and decreasing towards the anticentre. Applied to our case, the distances determined by assuming uniform absorption will appear to be too large if an increasing absorption coefficient is present, and too small in the opposite case. This effect on the distances is also true if the increase and decrease of the absorption coefficient is not a quality of a continuous stratum but the result of an accumulation or dispersal of absorbing clouds, or, finally, if it is the result of both of them. Thus, if the whole of the absorbing matter

* *M.N.*, 102, 205, 1942.

† *L.O.B.*, 14, 170, 1930 (Table 16).

has a relation to the structure of the galaxy the resulting differences in distances along the galactic circle will give the necessary indication.

Curve I, representing these differences, has three main features: it shows a prominent maximum, it is asymmetrical, and it shows some irregularities.

The position of the maximum is the diameter between 350° and 170° galactic longitude. This direction points undoubtedly towards the galactic centre, since it falls within the range of other determinations; among these Schilt's paper on star streams * and Rosenhagen's † on the K-term result in 351° and 350° respectively. It seems therefore

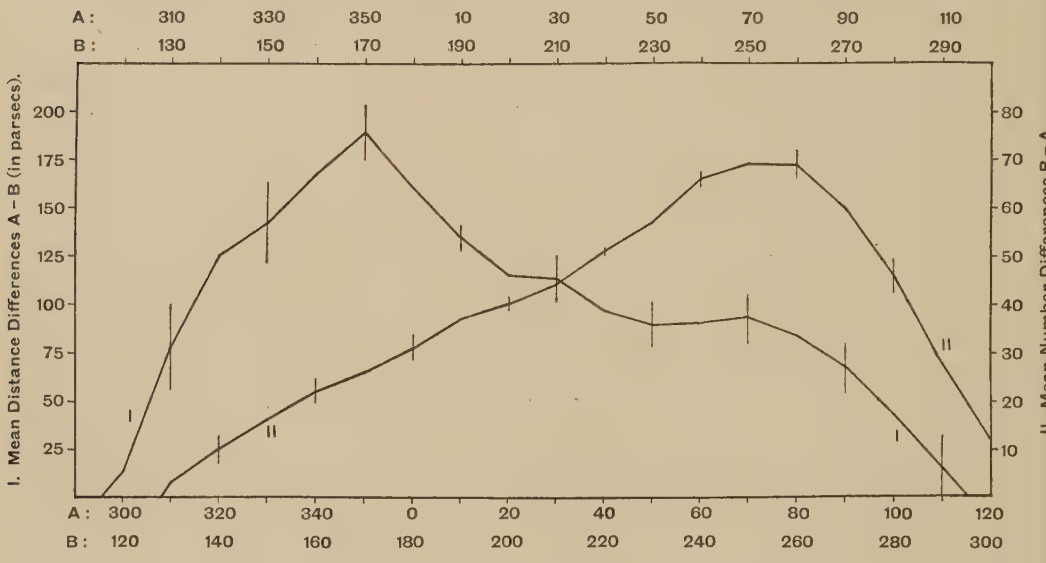

FIG. 1.—*Relation between Galactic Longitude and*
I. Mean Cluster Distances.
II. Mean Cluster Numbers.
The vertical lines indicate the probable error.

almost certain that the absorption is stronger towards the centre than in the opposite direction. This is confirmed by investigations on the space reddening of B stars by Stebbins, Huffer ‡ and Whitford § who find a conspicuous increase of reddening towards the centre and a lack of reddened stars near the anticentre. Zug's ‖ paper on colour excesses also shows, on the average, a decrease of the differential effect of absorption towards the anticentre.¶

The question of symmetry depends on the internal constitution of the absorbing matter. If we assume a continuous galactic stratum with an absorption coefficient regularly increasing towards the centre then the curve should be symmetrical. If this symmetry is partially or wholly missing then we must suppose that most or all of the change in the absorption coefficient is due to a peculiar distribution of absorbing clouds. Applied to our case the lack of symmetry indicates that the absorbing matter is composed of an irregular accumulation of absorbing clouds towards the centre within the galaxy which may or may not be bedded in a uniformly absorbing stratum.

Some of the more conspicuous irregularities of this distribution are shown in Curve I,

* *A.J.*, 39, 143, 1929.
† *A.N.*, 242, 401, 1931.
‡ *Publ. Washburn*, 15, 5, 247, 1934.
§ *Ap. J.*, 90, 224, 1939; 91, 49, 1940; 92, 194, 1940.
‖ *L.O.B.*, 16, 138, 1933.
¶ See also O'Keefe, *Ap. J.*, 94, 1941 (Plates XVIII and XIX).

and we can easily correlate them with the positions of absorbing clouds found in various papers. A survey of these results is given by Greenstein.* The extended region of Ophiuchus, Scorpio, Scutum, Serpens accounts for the cloud agglomeration at the galactic centre. The large hump between 60° and 90° corresponds to the clouds in Cepheus and Cassiopeia, and that at 30° to the Cygnus cloud. (The distance of the latter cloud (700 parsecs) is probably the reason why Stebbins and Huffer find, in discordance with Zug's larger absorption, a comparatively small colour excess. Their star-distances (average about 570 parsecs) are probably not much affected by this cloud. Even the extension of their investigation to more distant stars (average distance 670 parsecs) did not change the disparity.) The dip in the curve at 50° corresponds to the Vela cloud (230°), and the dip at 330° to the cloud in the Taurus–Orion–Auriga region (150°). All clouds listed in Greenstein's table have an image in our curve, and *vice versa*.

Curve II, representing the differences of cluster numbers between the disk halves (B – A), shows also a prominent maximum, it is also asymmetrical, but the irregularities are insignificant.

The position of the maximum at about 250° galactic longitude suggests an eccentric position of the Sun with regard to the cluster system in the opposite direction and it agrees with Trumpler's result (p. 181) placing the centre of the whole cluster system at 247°. The asymmetry of the curve is due to irregular cluster distribution along the galactic circle, as will also be seen in Trumpler's Table 18. No general influence of absorption on the cluster number seems detectable: around 350°/170°, where the difference in absorption is greatest, the differences in star numbers are nearly the same as around 115°/295° (no difference in absorption). Nor can any single sudden change in star numbers (*e.g.* the deep minimum between 0° and 50°, mentioned by Trumpler with regard to his fig. 2) suppress symmetry, although it impresses itself, in the procedure of smoothing, on the whole run of the curve It is, however, flattened out and its last residue can be seen as a very shallow irregularity between 0° and 50°.

4. *Conclusions.*—A random distribution of clusters throughout the whole galaxy would suggest a central position of the Sun in the observed cluster group (selected from all clusters only by the limits of observation). Even a change of absorption with galactic longitude will not affect this central position, as shown above. The observed distribution of cluster numbers must therefore be due to a true eccentric displacement of the Sun in a real cluster system which is not merely a selection from a general random distribution. This supports Trumpler's suggestion that the observed clusters are members of a cluster system situated in the so-called local star system (the centre of which is at about 250° longitude) and that the eccentricity of the Sun relative to this system is real. But the present data, incomplete on account of selection, do not permit a description of its internal constitution.

The asymmetrical distribution of absorption conditions in longitude and the presence of so-called galactic windows indicate that the galactic absorbing matter consists almost entirely of irregular absorbing clouds and that a continuous stratum could have only a very small absorption coefficient. It is therefore advisable to consider the "mean absorption" to be the "expectation of absorption" in a statistical sense, as Greenstein (p. 365) suggests. In fact, even this expectation depends on galactic longitude because the distribution of absorbing clouds is not a random one. It is found that there is a concentration of these clouds towards the galactic centre in the direction of 350° galactic longitude. If we derive the influence of this cloud distribution on the assumed mean absorption coefficient it can be shown that in the direction anticentre-centre the coefficient increases by about 5 per cent. per kpc. In any other direction this change is to be smaller.

* *H.A.*, 105, 360, 1936 (Table II).

Summary

A relation between observed distances of galactic clusters and galactic longitude is explained by rejecting the assumption of uniform absorption. It is shown that most of the absorption is due to irregular clouds and that these clouds accumulate towards the galactic centre at about 350° longitude. In this direction the absorption coefficient suffers an increase of 5 per cent. per kpc. A relation between cluster numbers and galactic longitude shows that the position of the Sun is eccentric to a local system, part of which seems to be the cluster system. Its centre is at about 250° longitude.

Norman Lockyer Observatory,
Sidmouth:
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NOTE ON THE CALCULATION OF PRECESSION

D. H. Sadler

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When accurate precessions and secular variations are not known, such as for instance in the formation of an observational catalogue, star positions are reduced from one equinox to another, either by means of tables, specially relating to the two epochs, or by direct calculation using the rigorous formulæ of spherical trigonometry. For stars near the pole, a high degree of formal accuracy is unobtainable by means of prepared tables, owing either to the failure of the expansions upon which they are based or to the limitations of tabular accuracy. Such tables are given annually in the *Nautical Almanac* * and the *Berliner Jahrbuch* † for reduction to the standard equinox of 1950·0.

The tables in the *Nautical Almanac*, which are based on Ristenpart's formulæ, are not rigorous for high declinations; the approximate numerical maxima of the theoretical errors (their actual magnitude depends on the right ascension) are:

$$\text{in R.A.} \quad 4.10^{-6}\, t^3 \tan^3 \delta \text{ in units of } 0^s{\cdot}001,$$
$$\text{in Dec.} \quad 3.10^{-6}\, t^3 \tan^2 \delta \text{ in units of } 0''{\cdot}01,$$

where t is the number of years to or from 1950·0. Further, the limitation of tabular accuracy, even in 1943 when additional figures have been added to two of the tabulated quantities (A_2 and D_1), implies that a formal accuracy of $0^s{\cdot}01$ in R.A. and $0''{\cdot}01$ in Dec. can only be obtained with declinations less than 84°. The theoretical errors can thus be used to impose a limit on t, above which the theoretical error exceeds the tabular error; this is effectively 12 years, both conditions giving identical results. The tables can be used equally well for reduction from 1950·0 as to 1950·0.

In the *Berliner Jahrbuch* the tables are based on Andoyer's formulæ and are rigorous. An accuracy of $0^s{\cdot}01$ in R.A. can be obtained only for declinations less than about 84°, owing to the limitation of tabular accuracy in the main tables; also for higher declinations one of the permanent tables (that for ΔC and P) has a limited range, and thereafter either reference is necessary to the more complete table by Peters and Pilowski ‡, or an auxiliary calculation must be made. The first limitation is independent of the interval t, but the limiting polar distance imposed by the second condition depends directly on t; for $t = 20$, the limiting declination is 87°·5.

Now that star catalogues are reduced to standard equinoxes at intervals of 25 years, it is clear that such tables should cater for intervals up to 20 years, to allow for an observational programme spread over several years. For stars of high declination, where it is desired to give at least a formal accuracy of $0^s{\cdot}01$ in R.A., the reduction for precession may thus have to be done by the rigorous trigonometrical formulæ. (It is usual in fundamental catalogues to give a formal accuracy of $0^s{\cdot}001$.) In such cases the spherical triangle involved is small and Legendre's theorem can be applied, leading to formulæ whose theoretical error *decreases* as the pole is approached. Legendre's theorem, used in geodetic calculations, states that a spherical triangle whose sides are small may be replaced by a plane triangle with equal sides and with angles decreased by one-third of the spherical excess (sum of the interior angles $- 180°$) of the original triangle; it does not seem to have been applied to astronomical triangles.

* *N.A.*, Tables IX and X, pp. 553–557, 1943.
† *B.J.*, pp. 274*–276*, 1940.
‡ *Veroff. des Astr. Rechen-Inst.*, Nr. 49, 1930.

With the usual notation * for the precessional constants, a direct application of Legendre's theorem, making an appropriate approximation for the spherical excess, leads to the following formulæ:—

Put

$$a_0 = \alpha_0 + \zeta_0, \qquad M = \zeta_0 + z,$$
$$d_0 = 90^\circ - \delta_0, \qquad d = 90^\circ - \delta,$$
$$e = \tfrac{1}{6}\theta \sin d_0 \sin \alpha$$
$$= 0{\cdot}00019392\ldots\theta'' d_0 \sin \alpha \text{ seconds of time,}$$

then

$$d'' \sin H = \theta'' \sin (a_0 - e),$$
$$d'' \cos H = d_0'' - \theta'' \cos (a_0 - e),$$
$$\alpha = \alpha_0 + M + H - 2e,$$

in which a double accent denotes that the quantity is measured in seconds of arc.

In the sense true minus calculated the principal error terms of the above formulæ are:

$$\Delta\alpha = +\tfrac{1}{72}\theta'' \sin^3 d_0 \sin \alpha, \qquad \Delta\delta = -\tfrac{1}{120}\theta'' \sin^4 d_0 \cos \alpha,$$

from which it is seen that for an interval of 20 years the residual error in R.A. is less than $0^s{\cdot}002$ provided the declination is greater than 80°; the error in declination is considerably smaller.

These formulæ are easier to use than the rigorous ones of spherical trigonometry, and involve little more work than that required in using prepared tables to their extreme limit. The constants ζ_0, z, M and $\sin\theta$ are tabulated, for reduction either to or from 1950·0, each year in the *Nautical Almanac*, and for the years 1800–1960 in the two volumes of *Planetary Co-ordinates*; simple expressions for them are given in the explanations or in Newcomb's *Precessional Constants*.

The calculation of e presents no difficulties, as its maximum value is less than $1^s{\cdot}000$; also six-figure tables are normally sufficiently accurate for the functions of $(a_0 - e)$. The derivation of d and H from $d \sin H$ and $d \cos H$ is a simple operation on a calculating machine, using d_0 as a first approximation to the square root of the sum of the squares.

These formulæ possess no particular merit, but do illustrate the possible application of Legendre's theorem to the solution of small astronomical triangles, when the parts cannot be treated as infinitesimal.

The above investigation arose in connection with the preparation of a Greenwich Catalogue based on observations from 1931 to 1938. I wish to thank the Astronomer Royal for pointing out the limitations of the reduction tables in the *Nautical Almanac* and for thus arousing my interest in the problem.

Summary

Formulæ, based on Legendre's theorem, are given for the reduction for precession of star positions from one equinox to another; these formulæ have the advantage that their accuracy increases as the pole is approached.

H.M. Nautical Almanac Office,
c/o Royal Observatory,
Greenwich.

* See Newcomb's *Spherical Astronomy*, p. 239, or *Nautical Almanac*, p. 681, 1941.